Dual norms of generalized group norms

Loïc Fagot-Bouquet

CEA, LIST, Vision and Content Engineering Laboratory,
Point Courrier 173, F-91191 Gif-sur-Yvette, France

March, 2016

This document determines the dual norm of a large class of group norms as stated in Proposition 4. In Section 3, this result is used in the specific case of weighted $l_{\infty,1}$ norms.

1 Generalized group norms

Let $E = \mathbb{R}^n$ be a finite-dimensional vector space with $n > 0$. Let $G$ be a partition of $\{1,...,n\}$ that groups the dimensions of $E$ into $k$ groups, $G = (G_i)_{1 \leq i \leq k}$.

Given $u \in E$ we denote by $u|_{G_i}$ the vector in $\mathbb{R}^{\chi(G_i)}$ derived from $u$ by removing all the coordinates not in $G_i$. For each group $G_i$, let $f_i$ be a function from $\mathbb{R}^{\chi(G_i)}$ to $\mathbb{R}$. Then we denote by $u^G_{(f_i)}$ the vector in $\mathbb{R}^k$ defined by

$$u^G_{(f_i)} = \begin{bmatrix} f_1(u|_{G_1}) \\ \vdots \\ f_k(u|_{G_k}) \end{bmatrix}.$$

Given a norm $\Omega$ over $\mathbb{R}^k$ and norms $\Omega_i$ over $\mathbb{R}^{\chi(G_i)}$ respectively, $||.||_{\Omega_i(\Omega_i)}$ is defined by

$$||u||_{\Omega_i(\Omega_i)} = \Omega(u^G_{(f_i)}) = \Omega(\begin{bmatrix} \Omega_1(u|_{G_1}) \\ \vdots \\ \Omega_k(u|_{G_k}) \end{bmatrix}).$$
Proposition 1. Given a partition $G = (G_i)_{1 \leq i \leq k}$ of $\{1, ..., n\}$ and applications $\Omega : \mathbb{R}^k \to \mathbb{R}$ and, for $i \in \{1, ..., k\}$, $\Omega_i : \mathbb{R}^{G_i} \to \mathbb{R}$ that satisfy

(i) $\Omega$ is a norm non-decreasing in its arguments over $(\mathbb{R}^+)^n$,

(ii) $\forall i \in \{1, ..., k\}$, $\Omega_i$ is a norm,

then $||.||_{\Omega_i, (\Omega_i)}$ is a norm.

Proof. $||.||_{\Omega_i, (\Omega_i)}$ has to satisfy the following properties

(i) $\forall u \in \mathbb{R}^n, \forall \lambda \in \mathbb{R}$, $||\lambda u||_{\Omega_i, (\Omega_i)} = |\lambda| ||u||_{\Omega_i, (\Omega_i)}$,

(ii) $\forall u \in \mathbb{R}^n, \forall v \in \mathbb{R}^n$, $||u + v||_{\Omega_i, (\Omega_i)} \leq ||u||_{\Omega_i, (\Omega_i)} + ||v||_{\Omega_i, (\Omega_i)}$,

(iii) if $||u||_{\Omega_i, (\Omega_i)} = 0$ then $u = 0$.

(i): Let $u \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$,

$$||\lambda u||_{\Omega_i, (\Omega_i)} = \Omega((\lambda u)_{(\Omega_i)}) .$$

We have

$$(\lambda u)_{(\Omega_i)} = \begin{bmatrix} \Omega_1(\lambda u|_{G_1}) \\ \vdots \\ \Omega_k(\lambda u|_{G_k}) \end{bmatrix} = \begin{bmatrix} |\lambda|\Omega_1(u|_{G_1}) \\ \vdots \\ |\lambda|\Omega_k(u|_{G_k}) \end{bmatrix} = |\lambda| u_{(\Omega_i)} ,$$

thus,

$$||\lambda u||_{\Omega_i, (\Omega_i)} = \Omega((\lambda u)_{(\Omega_i)}) = \Omega(|\lambda| u_{(\Omega_i)}) = |\lambda| \Omega(u_{(\Omega_i)}) = |\lambda| ||u||_{\Omega_i, (\Omega_i)} .$$

(ii): Let $u \in \mathbb{R}^n$ and $v \in \mathbb{R}^n$,

$$||u + v||_{\Omega_i, (\Omega_i)} = \Omega((u + v)_{(\Omega_i)}) .$$
We have
\[(u + v)^G_{(Ω_i)} = \begin{bmatrix}
Ω_1(u|G_1 + v|G_1) \\
\vdots \\
Ω_k(u|G_k + v|G_k)
\end{bmatrix},\]
and
\[
\begin{bmatrix}
Ω_1(u|G_1 + v|G_1) \\
\vdots \\
Ω_k(u|G_k + v|G_k)
\end{bmatrix} \preceq \begin{bmatrix}
Ω_1(u|G_1) + Ω_1(v|G_1) \\
\vdots \\
Ω_k(u|G_k) + Ω_k(v|G_k)
\end{bmatrix},
\]
and
\[
\begin{bmatrix}
Ω_1(u|G_1) + Ω_1(v|G_1) \\
\vdots \\
Ω_k(u|G_k) + Ω_k(v|G_k)
\end{bmatrix} = \begin{bmatrix}
Ω_1(u|G_1) \\
\vdots \\
Ω_k(u|G_k)
\end{bmatrix} + \begin{bmatrix}
Ω_1(v|G_1) \\
\vdots \\
Ω_k(v|G_k)
\end{bmatrix} = u^G_{(Ω_i)} + v^G_{(Ω_i)},
\]
leading to
\[(u + v)^G_{(Ω_i)} \preceq u^G_{(Ω_i)} + v^G_{(Ω_i)},\]
where \(\preceq\) stands for an element-wise comparison of two vectors. As we assume \(Ω\) to be non-decreasing in its arguments over \((\mathbb{R}^+)^n\), it follows that
\[Ω((u + v)^G_{(Ω_i)}) \leq Ω(u^G_{(Ω_i)} + v^G_{(Ω_i)}).\]
But we also have
\[Ω(u^G_{(Ω_i)} + v^G_{(Ω_i)}) \leq Ω(u^G_{(Ω_i)}) + Ω(v^G_{(Ω_i)}),\]
and thus,
\[Ω((u + v)^G_{(Ω_i)}) \leq Ω(u^G_{(Ω_i)}) + Ω(v^G_{(Ω_i)}),\]
which can be written as
\[||u + v||_{Ω,(Ω_i)} \leq ||u||_{Ω,(Ω_i)} + ||v||_{Ω,(Ω_i)}.\]
Let $u \in \mathbb{R}^n$ such that $||u||_{\Omega,(\Omega_i)} = 0$, we have

$$
\Omega(u^{G}_i) = 0 \implies u^{G}_i = 0 \\
\implies \forall i \in \{1, ..., k\}, \; \Omega_i(u|_{G_i}) = 0 \\
\implies \forall i \in \{1, ..., k\}, \; u|_{G_i} = 0.
$$

Thus,

$$
||u||_{\Omega,(\Omega_i)} = 0 \implies u = 0.
$$

\[ \square \]

## 2 Dual norms of generalized group norms

Given a norm $\Omega$ over $\mathbb{R}^n$, its dual norm $\Omega^*$ is defined by:

$$
\Omega^*(u) = \max_{v/\Omega(v) \leq 1} u^T v.
$$

**Proposition 2.** Let $\Omega$ be a norm over $\mathbb{R}^n$, then:

$$
\forall u \in \mathbb{R}^n, \forall v \in \mathbb{R}^n, \; u^T v \leq \Omega^*(u)\Omega(v).
$$

**Proof.** Let $u \in \mathbb{R}^n$ and $v \in \mathbb{R}^n$, we assume that $v \neq 0$ (this specific case is obvious). Then $\Omega(v/\Omega(v)) = 1$ and thus

$$
\frac{u^T v}{\Omega(v)} \leq \max_{v/\Omega(v) \leq 1} u^T v.
$$

As a consequence, $u^T \frac{v}{\Omega(v)} \leq \Omega^*(u)$ and finally

$$
u^T v \leq \Omega^*(u)\Omega(v).
$$

\[ \square \]

**Proposition 3.** Let $\Omega$ be a norm over $\mathbb{R}^n$ and $u \in \mathbb{R}^n$, then:

$$
\exists v \in \mathbb{R}^n : u^T v = \Omega^*(u) \; \text{and} \; \Omega(v) = 1.
$$
Proof. Let $B_\Omega = \{ v/\Omega(v) \leq 1 \} \subset \mathbb{R}^n$, we have
\[
\Omega^*(u) = \max_{v \in B_\Omega} u^T v = \max_{v \in B_\Omega} f_u(v),
\]
where $f_u(v) = u^T v$ is a continuous function with respect to $v$. Due to the finite dimension of $E$ and the equivalence of norms on $E$, $B_\Omega$ is closed and bounded and is therefore a compact set. As a consequence, $f_u$ attains its maximum on $B_\Omega$, leading to
\[
\exists w \in B_\Omega : u^T w = \max_{v \in B_\Omega} u^T v.
\]
We now show that $w$ can be chosen with $\Omega(w) = 1$. First of all, in the case $u = 0$ the function $f_u$ is constant and any $w \in B_\Omega$ with $\Omega(w) = 1$ maximizes $f_u$ on $B_\Omega$.

We assume $u \neq 0$, and consider $w \in B_\Omega$ that maximizes $f_u$ on $B_\Omega$. Then $u^T w \geq u^T \frac{w}{\Omega(w)} > 0$. Furthermore, $\frac{w}{\Omega(w)} \in B_\Omega$ implies
\[
u^T w \geq u^T \frac{w}{\Omega(w)},
\]
and thus $\Omega(w) \geq 1$ which leads to $\Omega(w) = 1$.

**Proposition 4.** Given a partition $G = (G_i)_{1 \leq i \leq k}$ of $\{1, \ldots, n\}$ and norms $\Omega, (\Omega_i)_{1 \leq i \leq k}$ chosen such that
(i) $\left\| \cdot \right\|_{\Omega_i, (\Omega_i)}$ is a norm,
(ii) $\Omega$ is invariant by component-wise absolute value,
then the dual norm of $\left\| \cdot \right\|_{\Omega_i, (\Omega_i)}$ satisfies
\[
\left\| \cdot \right\|_{\Omega_i, (\Omega_i)}^* = \left\| \cdot \right\|_{\Omega^*, (\Omega_i^*)}.
\]

Proof. Let $u \in \mathbb{R}^n$, we have
\[
\left\| u \right\|_{\Omega_i, (\Omega_i)}^* = \max_{v/\left\| v \right\|_{\Omega_i, (\Omega_i)} \leq 1} u^T v = \max_{v/\left\| v \right\|_{\Omega_i, (\Omega_i)} \leq 1} \sum_{1 \leq i \leq k} (u_{|G_i})^T v_{|G_i}.
\]

Using Proposition 2, we have for $1 \leq i \leq k$, $(u_{|G_i})^T v_{|G_i} \leq \Omega_i^*(u_{|G_i})\Omega_i(v_{|G_i})$. Thus,
\[
\left\| u \right\|_{\Omega_i, (\Omega_i)}^* = \max_{v/\left\| v \right\|_{\Omega_i, (\Omega_i)} \leq 1} \sum_{1 \leq i \leq k} (u_{|G_i})^T v_{|G_i}
\]
\[
\leq \max_{v/\left\| v \right\|_{\Omega_i, (\Omega_i)} \leq 1} \sum_{1 \leq i \leq k} \Omega_i^*(u_{|G_i})\Omega_i(v_{|G_i}).
\]
We can write
\[
\max_{v/\|v\|_{\Omega_i, (\Omega_i)}} \sum_{1 \leq i \leq k} \Omega_i^* (u_{|G_i}) \Omega_i (v_{|G_i}) = \max_{v/\Omega(u_{|G_i}) \leq 1} (u_{|G_i}^* G_i) v_{|G_i}^* G_i \leq \max_{w/\Omega(w) \leq 1} (u_{|G_i}^* G_i) w.
\]

As \( \max_{w/\Omega(w) \leq 1} (u_{|G_i}^* G_i) w = \Omega^* (u_{|G_i}^* G_i) \), we finally have
\[
\|u\|_{\Omega_i, (\Omega_i)} \leq \Omega^* (u_{|G_i}^* G_i)
\]

Using proposition 3, there exists \( \mu \in \mathbb{R}^k \) such that
\[
\Omega^* (u_{|G_i}^* G_i) = (u_{|G_i}^* G_i) \mu \text{ and } \Omega(\mu) = 1.
\]

For \( i \in \{1, \ldots, k\} \), there exists \( w_{|G_i} \in \mathbb{R}^{\chi(G_i)} \) such that
\[
\Omega_i^* (u_{|G_i}) = (u_{|G_i})^T w_{|G_i} \text{ and } \Omega_i (w_{|G_i}) = 1.
\]

We consider the vector \( v \in \mathbb{R}^n \) defined by
\[
\forall i \in \{1, \ldots, k\}, v_{|G_i} = \mu_i w_{|G_i}.
\]

Then we have
\[
\|v\|_{\Omega_i, (\Omega_i)} = \Omega (v_{|G_i}^* G_i) = \Omega \left( \begin{array}{c}
\Omega_1 (\mu_1 w_{|G_1}) \\
\vdots \\
\Omega_k (\mu_k w_{|G_k})
\end{array} \right) = \Omega \left( \begin{array}{c}
|\mu_1| \Omega_1 (w_{|G_1}) \\
\vdots \\
|\mu_k| \Omega_k (w_{|G_k})
\end{array} \right) = \Omega \left( \begin{array}{c}
|\mu_1| \\
\vdots \\
|\mu_k|
\end{array} \right),
\]

and since \( \Omega \) is invariant by component-wise absolute value,
\[
\|v\|_{\Omega_i, (\Omega_i)} = \Omega \left( \begin{array}{c}
|\mu_1| \\
\vdots \\
|\mu_k|
\end{array} \right) = \Omega(\mu) = 1.
\]

Furthermore,
\[
u^T v = \sum_{1 \leq i \leq k} (u_{|G_i})^T (v_{|G_i}) = \sum_{1 \leq i \leq k} \mu_i (u_{|G_i})^T (w_{|G_i}) = \sum_{1 \leq i \leq k} \mu_i \Omega_i^* (u_{|G_i}).
\]
leading to
\[ u^T v = \sum_{1 \leq i \leq k} \mu_i \Omega^*_G(u|_{G_i}) = \mu^T(u^G_{(\Omega^*_i)}) = \Omega^*(u^G_{(\Omega^*_i)}). \]

As a consequence,
\[ ||u||^*_\Omega,_{(\Omega_i)} = \max_{v/||v||_{\Omega,_{(\Omega_i)}} \leq 1} u^T v \geq \Omega^*(u^G_{(\Omega^*_i)}), \]
and finally,
\[ ||u||^*_\Omega,_{(\Omega_i)} = \Omega^*(u^G_{(\Omega^*_i)}) = ||u||^*_{\Omega^*,_{(\Omega^*_i)}}. \]

\[ \square \]

3 Application to weighted \( l_{\infty,1} \) norms

The results of the two last sections are used here in the case of weighted \( l_{\infty,1} \) norms.

**Proposition 5.** Let \( \Omega \) be a norm over \( \mathbb{R}^n \) and \( \lambda \in \mathbb{R}^+ \), then
\[ (\lambda \Omega)^* = \frac{1}{\lambda} \Omega^*. \]

**Proof.** Let \( u \in \mathbb{R}^n \),
\[ (\lambda \Omega)^*(u) = \frac{1}{\lambda} \Omega^* = \max_{v/\lambda \Omega(v) \leq 1} u^T v \]
\[ = \max_{v/\Omega(v) \leq 1} \frac{1}{\lambda} u^T \lambda v \]
\[ = \frac{1}{\lambda} \max_{w=\lambda v} \Omega^*(w) \]
\[ = \frac{1}{\lambda} \Omega^*(u). \]

\[ \square \]
Proposition 6. Let \((w_i)_{1 \leq i \leq k} \in (\mathbb{R}^+)^k\) be some positive weights and \(G = (G_i)_{1 \leq i \leq k}\) a partition of \(\{1, \ldots, n\}\). The function \(l^w_{\infty,1}\), or \(\|\cdot\|_{\infty,1}^w\) is defined by

\[
\forall u \in \mathbb{R}^n, \quad \|u\|_{\infty,1}^w = \max_{1 \leq i \leq k} w_i \|u|_{G_i}\|_1,
\]

while the function \(l^\frac{1}{\infty,1}\) or \(\|\cdot\|_{1,\infty}^{\frac{1}{w}}\) is defined by

\[
\forall u \in \mathbb{R}^n, \quad \|u\|_{1,\infty}^{\frac{1}{w}} = \sum_{1 \leq i \leq k} \frac{1}{w_i} \|u|_{G_i}\|_\infty.
\]

Then \(\|\cdot\|_{\infty,1}^w\) is a norm over \(\mathbb{R}^n\) and its dual norm satisfies

\[
\|\cdot\|_{\infty,1}^w = \|\cdot\|_{1,\infty}^{w}.
\]

Proof. We have

\[
\|\cdot\|_{\infty,1}^w = \|\cdot\|_{1,\infty}^{(w_i l_1)}.
\]

The \(l_{\infty}\) norm is non-decreasing in its arguments over \((\mathbb{R}^+)^n\) and for each \(i \in \{1, \ldots, k\}\), \(w_i l_1\) is a valid norm. Therefore, Proposition 1 indicates that \(\|\cdot\|_{\infty,1}^w\) is a norm.

As the \(l_{\infty}\) norm is invariant by component-wise absolute value, Proposition 4 can be applied and gives

\[
\|\cdot\|_{\infty,1}^{w} = \|\cdot\|_{1,\infty}^{l_\infty,(w_i l_1)} = \|\cdot\|_{1,\infty}^{l_\infty,(\frac{1}{w_i} l_1^*)}.
\]

The dual norms of the \(l_1\) and \(l_{\infty}\) norms are given by \(l_1^* = l_{\infty}\) and \(l_\infty^* = l_1 [1]\), leading to

\[
\|\cdot\|_{1,\infty}^{w} = \|\cdot\|_{1,\infty}^{l_\infty,(\frac{1}{w_i} l_1^*)} = \|\cdot\|_{1,\infty}^{l_1,(\frac{1}{w_i} l_\infty)}.
\]

Thus,

\[
\|\cdot\|_{\infty,1}^{w} = \|\cdot\|_{1,\infty}^{\frac{1}{w}}.
\]

\(\square\)

References