Theorem 1 For the reduced identifiability matrices in the same group, let $N_M$ denote the number of matrices, let $N_P$ denote the number of unknown parameters, and let $N_{\text{max}}$ be the maximum number of the “1” elements in one row of all the matrices.

- When $N_P > N_M$, all the parameters in the same group are unidentifiable;
- When $N_P = N_M$, the parameters are globally identifiable if $N_{\text{max}} = 1$, and locally identifiable if $N_{\text{max}} > 1$;
- When $N_P < N_M$, the parameters are at least locally identifiable.

Proof. By definition of structural identifiability, the number of solutions to the identifiability equations is the key to verify parameter identifiability. According to the generation rule of identifiability matrices, $N_M$ identifiability matrices correspond to $N_M$ different identifiability equations. If the number of matrices is less than the number of unknown parameters (that is, $N_M < N_P$), the number of polynomial symbolic equations is then less than the number of unknown parameters. Because the identifiability equations are in the form of $\sigma_y = \sum \prod_{\text{path edge}} \theta_i$ and the order of $\theta_i$ is at most one in each monomial, this leads to an underdetermined system such that there exist an infinite number of solutions. Therefore, all the parameters are unidentifiable if the number of matrices is less than the number of parameters.

If the number of matrices is equal to the number of parameters in a group (i.e., $N_M = N_P$), then we have $N_P$ polynomial equations for real/complex variables with
unknown parameters. According to the work of Garcia and Li [1], the number of solutions is equal to 
\[ q = \prod_{i=1}^{N_p} q_i, \]
where \( q_i \) is the degree (the power of the highest order term) of equation \( i \). If the maximum number \( N_{\text{max}} \) of the “1” elements in one row of all the matrices is equal to 1, then \( q = 1 \); this means that each parameter has a unique solution. In other words, every parameter is globally identifiable in this group. If the number \( N_{\text{max}} \) is greater than 1, then \( q > 1 \); this means there exist a finite number of solutions. In other words, every parameter is locally identifiable in the same group.

If the number of matrices is greater than the number of parameters in a group (i.e., \( N_p < N_M \)), then the equations corresponding to the matrices form an overdetermined polynomial system. For such a system, the number of solutions cannot be determined because even if we only change the constants in the system (e.g., constant coefficients), the number of solutions can be zero, one or a finite number, which needs to be analyzed case by case. However, from the parameter estimation point of view (i.e., the original problem becomes an optimization problem), there exist multiple local solutions [2-5]. Therefore, when \( N_p < N_M \), the parameters are at least locally identifiable.

References