Bayesian inference and Gibbs sampling.

The final representation of our genotype-phenotype SEM model is:

\[
\eta = B\eta + \Pi \bar{g} + \varepsilon
\]

\[
\begin{pmatrix}
\vec{u} \\
\vec{v}
\end{pmatrix} = \Lambda \eta + K\bar{y} + \delta,
\]

where latent variables \(\bar{g}, \bar{y}\) and \(\bar{v}\) mimic ordinal observed variables \(g, y\) and \(v\). Let the size of the dataset be \(n\) and it contains matrices of observations for \(g, y\) and \(v\): \(G\) of size \((n_g \times n)\), \(Y\) of size \((n_y \times n)\) and \(V\) of size \((n_v \times n)\), respectively. At each iteration of an MCMC method, we generate matrices of observation for respective latent variables - \(\bar{g}\) for \(g\), \(\bar{y}\) for \(y\) and \(\bar{v}\) for \(v\) - by the same way. For example, the element in \(\bar{g}\) at \(i\)-th row and \(j\)-column is randomly drawn from standard normal distribution truncated to the support \(s(G, i, j) = (l(G, i, j), r(G, i, j))\):

\[
l(G, i, j) = Q\left(\frac{1}{n} \sum_{k=1}^{n} [G_{ik} < G_{ij}]\right), r(G, i, j) = Q\left(\frac{1}{n} \sum_{k=1}^{n} [G_{ik} \leq G_{ij}]\right),
\]

where \(Q(\cdot)\) is the quantile function associated with the standard normal distribution, \([\ldots]\) are the Iverson brackets (Knuth 1992). Thereby, we generated elements in \(\bar{g}, \bar{y}\) and \(\bar{v}\) form truncated normal distributions independently from other parameters and variables in the model:

(i) \(\forall (i, j) \in \{1 \ldots n_g\} \times \{1 \ldots n\}\): generate \(\bar{g}_{ij}\) from \(t\mathcal{r}\mathcal{N}(0, 1, s(G, i, j))\);

(ii) \(\forall (i, j) \in \{1 \ldots n_y\} \times \{1 \ldots n\}\): generate \(\bar{y}_{ij}\) from \(t\mathcal{r}\mathcal{N}(0, 1, s(Y, i, j))\);

(iii) \(\forall (i, j) \in \{1 \ldots n_v\} \times \{1 \ldots n\}\): generate \(\bar{v}_{ij}\) from \(t\mathcal{r}\mathcal{N}(0, 1, s(V, i, j))\).

To obtain the posterior distribution for latent variables \(\eta\), we applied the approach proposed in (Lee 2007) at p.83. Then, for \(j\)-th sample, we denote the values of latent variables \(\eta\) as \(H_j\) and draw it from multivariate normal distribution:

(iv) \(\forall j \in \{1 \ldots n\}\): generate \(H_j\) from \(m\mathcal{v}\mathcal{N}((\Sigma^{-1} + \Lambda^t \Theta^{-1} \Lambda) \Lambda^t \Theta^{-1} \bar{P}_j, (\Sigma^{-1} + \Lambda^t \Theta^{-1} \Lambda))\), where \(\Sigma = C(\Pi \Pi^t + \Theta_\varepsilon) C^t\), \(C = (I - B)^{-1}\) and \(\bar{P}_j = (U_j, \bar{V}_j)^t - K\bar{G}_j\)

Parameters in the structural part

We performed the Bayesian inference of posterior distributions for parameters in the structural part \((B, \Pi, \Theta_\varepsilon)\) similarly to (Lee 2007) at p.84. Let consider an equation in the structural part corresponding to \(i\)-th latent variable \(\eta_i\):

\[
\eta_i = \Gamma_i \omega_i + \varepsilon_i,
\]

where \(\Gamma_i\) is a set of parameters in \(i\)-th rows of \(B\) and \(\Pi\); \(\omega_i\) contains subsets of \(\eta\) and \(\bar{g}\) corresponding to positions of parameters in \(i\)-th rows of \(B\) and \(\Pi\); \(\varepsilon_i\) is \(i\)-th component in the vector of random errors and it is normally distributed with zero mean and variance equal to \(\Theta_{\varepsilon i}\). We set the inverse Gamma and multivariate normal prior distributions for \(\Theta_{\varepsilon i}\) and \(\Gamma_i\) respectively:

\[
\Theta_{\varepsilon i} \sim \mathcal{IG}(\alpha_{\Theta i}, \beta_{\Theta i}),
\]

\[
\Gamma_i \sim \mathcal{MN}(0, \nu, \Sigma_{\Gamma i}),
\]
where \( F_{0i} \) is the ML estimate for \( \Gamma_i \), \( \Phi_{0i} \) is the Identity matrix; parameters for inverse Gamma distribution are taken as in (Lee 2007) at p.76: \( \alpha_{0ei} = 9, \beta_{0ei} = 4 \). Let the dataset for \( \omega_i \) be the matrix \( \Omega_i \) of size \( (n_{oi} \times n) \); the dataset for \( \eta_i \) be the horizontal vector \( H_i \) of length \( n \). During the Bayesian inference of posterior distributions, we obtained the following parameters of posterior distributions:

\[
\Phi_i = (\Phi_{0i}^{-1} + \Omega_i\Omega_i^T)^{-1},
\]
\[
F_i = \Phi_i[\Phi_{0i}^{-1}F_{0i} + \Omega_iH_i^T],
\]
\[
\alpha_{ei} = \alpha_{0ei} + n/2,
\]
\[
\beta_{ei} = \beta_{0ei} + \frac{1}{2}[H_iH_i^T + F_{0i}^T\Phi_{0i}^{-1}F_{0i} - F_i^T\Phi_i^{-1}F_i].
\]

Then, values of \( \Gamma_i \) and \( \Theta_{ei} \) are drawn in the following order:

(v) generate \( \Theta_{ei} \) from \( IG(\alpha_{ei}, \beta_{ei}) \),
(vi) generate \( \Gamma_i \) from \( \mathcal{MN}(F_{0i}, \Theta_{ei} \Phi_i) \).

**Parameters in the measurement part**

As for the structural part, we performed the Bayesian inference of posterior distributions for parameters in the measurement part \( (\Lambda, K, \Theta_{\tilde{g}}) \) similarly to (Lee 2007) at p.84. Let consider an equation in the measurement part corresponding to \( i \)-th phenotype \( p_i \):

\[
p_i = A_i w_i + \delta_i,
\]

where \( A_i \) is a set of parameters in \( i \)-th rows of \( \Lambda \) and \( K \); \( w_i \) contains subsets of \( \eta \) and \( \tilde{y} \) corresponding to positions of parameters in \( i \)-th rows of \( \Lambda \) and \( K \); \( \delta_i \) is \( i \)-th component in the vector of random errors and it is normally distributed with zero mean and variance equal to \( \theta_{\delta i} \). We set the inverse Gamma and multivariate normal prior distributions for \( \theta_{\delta i} \) and \( A_i \) respectively

\[
\theta_{\delta i} \sim IG(\alpha_{\delta i}, \beta_{\delta i}),
\]
\[
A_i \sim \mathcal{MN}(D_{0i}, \theta_{\delta i} \Psi_{0i})
\]

where \( D_{0i} \) is the ML estimate for \( A_i \), \( \Psi_{0i} \) is the Identity matrix; parameters for inverse Gamma distribution are taken as in (Lee 2007) at p.76: \( \alpha_{\delta i} = 9, \beta_{\delta i} = 4 \). Let the dataset for \( w_i \) be the matrix \( W_i \) of size \( (n_{wi} \times n) \); the dataset for \( p_i \) be the horizontal vector \( P_i \) of length \( n \). During the Bayesian inference of posterior distributions, we obtained the following parameters of posterior distributions:
\[ \psi_i = (\psi_{0i}^{-1} + W_i W_i^\top)^{-1}, \]
\[ D_i = \psi_i [\psi_{0i}^{-1} D_{0i} + W_i P_i^\top], \]
\[ \alpha_{\delta i} = \alpha_{0\delta i} + n/2, \]
\[ \beta_{\delta i} = \beta_{0\delta i} + \frac{1}{2} \left[ P_i P_i^\top + D_{0i} \psi_{0i}^{-1} D_{0i} - D_i \psi_i^{-1} D_i \right]. \]

Then, values of $A_i$ and $\theta_{\delta i}$ are drawn in the following order:

(vii) generate $\theta_{\delta i}$ form $\mathcal{J}(\alpha_{\delta i}, \beta_{\delta i})$,
(viii) generate $A_i$ form $\mathcal{M}(D_i, \theta_{\delta i} \psi_i)$.

Scripts are available at https://github.com/iganna/mtmlsem.