
Genotypes are coded using the standardized $x_{im} = (a_{im} - 2p_m)/\sigma_m$. The statistical model is $y = X\beta + e$ and the aim is to predict $g = W\beta$. Marker effects are distributed in $\mathcal{L}(0, \mathbf{I}\sigma_\beta^2)$. The $y$ distribution conditional to $X$ is such that $E(y|X) = 0$ and $\nu(y|X) = XX'\sigma_\beta^2 + \mathbf{I}\sigma_e^2$. The total phenotypic variance is $\nu(y) = \nu_x[E(y|X)] + \nu_x[\nu(y|X)] = \nu_x[\nu(y|X)] = n_M E[G]\sigma_\beta^2 + \mathbf{I}\sigma_e^2$, where $G$ is the genomic matrix. The markers BLUP is $\hat{\beta} = (X'X + \mathbf{I}_\lambda)^{-1}X'y = Py$ and the GEBVs are $\hat{g} = WPy = Sy$.

Let $PX = T = T'$ giving $P(XX' + \mathbf{I}_\lambda)P' = X'X(X'X + \mathbf{I}_\lambda)^{-1} = T = I - \lambda (X'X + \mathbf{I}_\lambda)^{-1}$

We look for $E[r^2] = \frac{\text{cov}^2(g, \hat{g})}{\nu(g)\nu(\hat{g})}$

$v(g) = v(W\beta) = E_w[v_\beta(W\beta|W)] + E_{w}[E_\beta(W\beta|W)] = E_w[v_\beta(W\beta|W)] = E_w[W\beta\sigma_\beta^2W']$

Thus $v(g) = E[W] I\sigma_\beta^2 E[W] + \text{tr}[I\sigma_\beta^2 v(W)] = \sigma_\beta^2 \sum_{m=1}^{n_M} \sigma_m^2$.

$v(g) = E_{wx}[v(\hat{g}|W, X)] + E_{wx}[E(\hat{g}|W, X)] = E_{wx}[v(\hat{g}|W, X)]$

$v(g|W, X) = WP(XX'\sigma_\beta^2 + \mathbf{I}\sigma_e^2) P' = WT\sigma_\beta^2$

$E_{wx}[v(\hat{g}|W, X)] = \sigma_\beta^2 E_X[E_w[WTW'|X]]$

Let $\varphi_w = E[W|X]$, we get $E_{wx}[v(\hat{g}|W, X)] = \sigma_\beta^2 E_X[\varphi_w T\varphi_w' + \text{tr}(T\nu(W|X))]

v(g) = \sigma_\beta^2 (E_X[\varphi_w T\varphi_w'] + E_X[\text{tr}(TD_{WX})])$

$\text{cov}(g, \hat{g}) = E_{wx}[\text{cov}(v(g, \hat{g}|W, X)) + \text{cov}(v(g|W, X), v(\hat{g}|W, X)) = E_{wx}[\text{cov}(v(g, \hat{g}|W, X)]$

$\text{cov}(g, \hat{g}|W, X) = \text{cov}(W\beta, W(X'X + \mathbf{I}_\lambda)^{-1}X'(X\beta + e)|W, X)$

$\text{cov}(g, \hat{g}|W, X) = \text{cov}(W\beta, WT\beta|W, X) = WT\sigma_\beta^2$

Thus $\text{cov}(g, \hat{g}) = v(g)$

and $E[r^2] = \frac{v(g)}{v(\hat{g})} = \frac{\sigma_\beta^2 (E_X[\varphi_w T\varphi_w'] + E_X[\text{tr}(TD_{WX})])}{\sigma_\beta^2 \sum_{m=1}^{n_M} \sigma_m^2}$

If we now suppose that

- Individuals are unrelated $\varphi_w = E_W[W|X] = E_W[W] = 0$ et $D_{W|X} = D_W$
- Markers are in L.E. $\nu(W) = D_W = I$
- $X'X \sim E[X'X] = n_R I$

$E[r^2] = \frac{Ex[\text{tr}(T)]}{n_M}$ with $Ex[\text{tr}(T)] = Ex[n_M - \lambda \frac{n_M}{n_R + \lambda_R}] = \frac{n_R n_M}{n_R + \lambda_R}$

Finally $E[r^2] = \frac{n_R}{n_R + \lambda_R} = \frac{n_R}{n_R + n_M \lambda} = \frac{n_R h^2}{n_M h^2 + 1 - h^2}$