1 Rates in ordinary differential equations

All models have biases. The simplest model is the correlation between two variables, where our interpretation decides if variable $x$ is influencing variable $y$, or variable $y$ is influencing variable $x$, or the two variables by coincidence vary in the same place. In a dynamical model, we write relationships as mathematical equations. As an example, we could use the development of a mosquito from pupa to adult. In real life, such a metamorphosis could be described by delay differential equations (equation 1), but for practical purposes they are often approximated and written as ordinary differential equations (ODEs, equation 2).

$$\frac{dP(t)}{dt} = -P(t - \tau) \quad (1)$$

where $\tau$ is the number of days required to develop from pupa, $P$, to adult, $A$.

$$\frac{dA(t)}{dt} = P(t - \tau)$$

where $A$ is the number of adults, $P$ is the number of pupae, and $r$ is the development rate from pupa to adult.

By deciding to use ODEs, we have introduced the first error. ODEs are capable of producing half a pupa and half a mosquito at any given time, and such pupae would converge towards zero. Let us continue the following example. We start with two pupae, $P = 2$, and zero adults $A = 0$, neglecting mortality. Development from pupa to adult takes 2 days. The exact solution of this problem would be

$P(t = 0) = 2, P(t = 1) = 2, P(t = 2) = 0$ and
$A(t = 0) = 0, A(t = 1) = 0, A(t = 2) = 2$.

In the framework of ODEs, the value of $r$ would decide how fast development occurs. One method is to define the rate as per day, $\text{day}^{-1}$. In this case, $r = 1/2$. The exact solution using ODEs then becomes

$P(t = 0) = 2 \cdot e^{-r \cdot t} = 2.00, P(t = 1) = 1.21, P(t = 2) = 0.74$, and
$A(t = 0) = 2 - 2 \cdot e^{-r \cdot t} = 0, A(t = 1) = 0.79, A(t = 2) = 1.26$.

Another way to define the development rate, $r$, is to consider the fraction of pupae that have developed at time $t$. Let us say that 50% of the pupae had developed into adults by the second day ($t = 1$); we could then find an exact solution that satisfies this condition:
\[ P(t = 1) = P(t = 0) \cdot e^{-r \cdot 1} = P(t = 0) \cdot 0.5 \]
\[ e^{-r} = 0.5 \]
\[ r = -\log(0.5) \]  \hspace{1cm} (3)

This approach then defines the development rate as the time it takes for 50\% of the mosquitoes to develop from pupae to adults, \( d(t) \), or more generally, \( r = -\log(0.5) \cdot d(t)^{-1} \).