Figure 1: The configuration of a sensor array

where the superscript $[\cdot]^T$ denotes the matrix transpose operation. Next we introduce the mode vector $a_m(\theta_l) \ (m = 1, 2, \cdots, M)$, where $\theta_l$ is the angle of the $l$th wave. $a_m(\theta_l)$ is defined as:

$$a_m(\theta_l) = [\exp(jm\Psi_{m,1}(\theta_l)), \cdots, \exp(jm\Psi_{m,K}(\theta_l))]^T.$$  

$\Psi_{m,k}(\theta_l) \ (k = 1, 2, \cdots, K)$ is the phase of the $l$th wave of the $m$th frequency at the $k$th sensor and is expressed as:

$$\Psi_{m,k}(\theta_l) = -2\pi f_m \frac{d_k \sin \theta_l}{c},$$

where $c$ is the velocity of sound, $f_m$ is the $m$th frequency, and $d_k$ is the distance between the $k$th sensor and the 1st sensor (as shown in Fig. 1). Assuming that the receiving waves are plane waves, $S_m$ can be written as:

$$S_m = A_m F + N,$$

where $A_m$ is:

$$A_m = [a_m(\theta_1), \cdots, a_m(\theta_L)]$$

$F = [F_1, \cdots, F_L]^T$,

and $N$ is the Gaussian noise vector with zero means and equal variances $\sigma^2$.

Similarly, we introduce a new mode vector $g_k(\tau_l) \ (k = 1, 2, \cdots, K)$, where $\tau_l$ is the propagation delay time of the $l$th wave. $g_k$ is defined as:

$$g_k(\tau_l) = [\exp(-j2\pi f_1 \tau_l), \cdots, \exp(-j2\pi f_M \tau_l)]^T.$$  

By using $g_k$, $T_k$ can be written as:

$$T_k = G_k F + N,$$

where $G_k$ is:

$$G_k = [g_k(\tau_1), \cdots, g_k(\tau_L)].$$

2.2 The S-MUSIC Algorithm

The S-MUSIC algorithm is for DOA estimation. This algorithm is used with single-frequency signals. By equation (1), the correlation matrix calculated using $S_m$ is given as:

$$R_{ss} = E\left[ S_m S_m^H \right] = A_m E\left[ F F^H \right] A_m^H + E\left[ N N^H \right] = A_m A_m^H + \sigma^2 I \left( \alpha \equiv E\left[ F F^H \right] \right),$$

where $A_m = \begin{bmatrix} a_m(\theta_1) & \cdots & a_m(\theta_L) \end{bmatrix}$. 
