foundation by using Banach fixed point theorem. In fact, this restriction on the candidate space for solutions is justified by physical considerations.

The rest of the paper is organized as follows: In Section 2, we describe our problem in detail, and formulate an integral equation equivalent to the non-linear and non-uniform beam equation. The properties of the non-linear, non-uniform elastic foundation are analyzed in Section 3, and a close investigation on the basic integral operator $K$, which has an important role in both linear and non-linear beam equations, is performed in Section 4. In Section 5, we define the subspace on which our integral operator $\Psi$ becomes a contraction, and show the existence and the uniqueness of the solution in this space. Finally, Section 6 recapitulates the overall procedure of the paper, and explains some of the intuitions behind our formulation for the reader.

2 Definition of The Problem

We deal with the question of existence and uniqueness of solutions of non-linear deflections for an infinitely long beam resting on a non-linear elastic foundation which is non-uniform in $x$. Figure 1 shows that the vertical deflection of the beam $u(x)$ results from the net load distribution $p(x)$:

$$p(x) = w(x) - f(u, x).$$  (1)

In (1), the two variable function $f(u, x)$ is the non-linear spring force upward, which depends not only on the beam deflection $u$ but also on the position $x$, and $w(x)$ denotes the applied loading downward. For simplicity, the weight of the beam is neglected. In fact, the weight of the beam could be incorporated in our static beam deflection problem by adding $m(x)g$ to the loading $w(x)$, where $m(x)$ is the length-wise mass density of the beam in $x$-coordinate and $g$ is the gravitational acceleration. The term $m(x)g$ also