In this section we will therefore first provide theoretical lower and upper bounds on the optimal power-rate trade-off for a network without interference. It is then discussed how these bounds extend to the interference case. Based on these insights we assess the price of greening for different practical settings by numerical simulations using practical channel models. This will allow to identify the typical trends that can be expected in DSL access systems under different practical settings.

1) Theoretical analysis of power-rate trade-off:
We start with a multi-tone single-user bound analysis:

**Lemma IV.1.** For a single-user DSL system, the relative reduction in optimal data rate for a given relative reduction in transmit power of factor \( c \), i.e., \( f(c) \), is given by the expression:

\[
 f(c) = \frac{R^n(c \cdot P^{n,\text{tot}})}{R^n(P^{n,\text{tot}})}
\]

with \( R^n(P) \) being the optimal data rate for given total power \( P \) and \( \text{CNR}_k^n = |h_k|_2^2 / (1/s_k^n) \).

**Proof:** See Appendix B.

Lemma IV.1 shows how the power-rate trade-off, i.e., \( f(c) \) in function of \( c \), depends on the CNRs, the transmit power budget, and the power saving factor \( c \). Using this lemma we can obtain the following theorem describing the theoretical bounds on the power-rate trade-off.

**Theorem IV.1.** The power-rate trade-off, i.e., \( f(c) \) in function of \( c \), for a single-user DSL system is in the worst case linear. This linear relation is attained if all channel-to-noise ratios (CNRs) or the transmit power budget \( P^{n,\text{tot}} \) go to zero. In contrary, when the CNRs or \( P^{n,\text{tot}} \) go to infinity, the trade-off vanishes. Formally:

\[
\begin{align*}
\lim_{k \in K, \text{CNR}_k^n \to 0} & f(c) = c, \quad \text{and} \quad \lim_{p^{n,\text{tot}} \to 0} f(c) = c, \quad \text{and} \quad \lim_{p^{n,\text{tot}} \to \infty} f(c) = 1, \\
\lim_{k \in K, \text{CNR}_k^n \to \infty} f(c) = 1, \quad \text{and} \quad \lim_{p^{n,\text{tot}} \to \infty} f(c) = 1.
\end{align*}
\]

**Proof:** The proof follows trivially from Lemma IV.1 by taking a limit operation of (16).

Lemma IV.1 and Theorem IV.1 provide lower and upper bounds on the optimal power-rate trade-off that hold for any single-user DSL system setting. Furthermore they show that the trade-off depends on the CNRs and the transmit power budget: larger CNRs and a larger transmit power budget result in a more favorable power-rate trade-off, as more power can be saved for a certain data rate decrease.

When extending these bounds to the multi-user interference case, one would at first sight expect that adding crosstalk to the noise reduces the overall CNRs and thus results in a less favorable power-rate trade-off, i.e., being more linear. However, crosstalk interference also results in a data rate decrease with

\[\text{Note that for the multi-user setting, we will use CNR to refer to the channel to interference and noise ratio, i.e., } \text{CNR}_k^n = |h_k|_2^2 / (\sum_{m \neq n} |h_m|_2^2 + \sigma_k^n)\]