B. Bi-Orthonormal Modulation is UMP

According to the results from the preceding section, we see that UMP property does not require an accurate complex orthonormal alphabet. Inspired also by [10] we have a conjecture that bi-orthonormal modulation is UMP. The Appendix proves the following lemma.

Lemma 15. Bi-orthonormal modulation is UMP.

Remark 16. Interestingly, the symmetrical XC matrix with the same main diagonal is not sufficient in this case and an extra kind of symmetry, which obeys XOR as well, is required.

C. UMP-CPM Design

1) CPM basic properties: CPM is a constant envelope modulation (suitable e.g. for satellite communication) with more compact spectrum in compare to the linear modulations with constant envelope (with rectangular (REC) modulation pulse). It has a multidimensional alphabet and better spectral properties than FSK (no Dirac pulses in the spectrum and faster asymptotic spectrum attenuation due to the continuous phase). Bandwidth requirements of the considered schemes are investigated in Sec. VI-C. CPM includes memory [21] and its modulator consists of the discrete part including memory and the non-linear memoryless part [22]. Denominator of CPM modulation index \( \kappa \) is proportional to the number of modulator states described by its trellis and the optimal decoder need to perform Viterbi decoding.

CPM possess several degrees of freedom, for simplicity, we restrict on the full-response (i.e. the frequency pulse is of the symbol length) and minimum shift \( \kappa = 1/2 \) case for which constellation space alphabet is \( N = M \) dimensional.

2) Design of full-response \( \kappa = 1/2 \) UMP-CPM by Pulse Shape Optimization: In the same way we have excluded channel coding from design of UMP alphabet, we do not need to consider modulation memory of CPM. In our case, the non-linear memoryless part is determining.

Assumed full-response \( \kappa = 1/2 \) CPM has the modulation trellis with only two states and the non-linear memoryless alphabet consists of \( 2M \) signals of which the first half starting from the first state have opposite sign than the other half starting from the latter state, see e.g. the trellis of binary scheme in Fig 8.

Our design is based on Lemma 15, utilizing the above mentioned symmetries, we design a bi-orthonormal UMP modulation simply keeping orthonormal signals starting from the first state.

Figure 8: Binary full-response \( \kappa = 1/2 \) CPM trellis. Note, if signal space vectors \( s_0 \) and \( s_1 \) are orthonormal, than the resulting alphabet is bi-orthonormal.

3) CPM Signals Notation: Let us denote positive-sign alphabet (signals starting from the zero state) \( \mathcal{A}^+ \) and negative-sign alphabet \( \mathcal{A}^- = -\mathcal{A}^+ \). The overall alphabet (non-linear memoryless part) is \( \mathcal{A} = \{ \mathcal{A}^+, \mathcal{A}^- \} \). Assuming unit energy signals, full-response \( h = 1/2 \) CPM has

\[
\mathcal{A}^+ = \{ s_i(t) \}_{i=0}^{M-1} = \left\{ e^{j\pi (\frac{t}{2} + \beta(i))} \right\},
\]

where data symbol \( c \in \left\{ -(M_e - 1), -(M_e - 3), \ldots, (M_e - 1) \right\} \), \( t \) is normalized to one symbol duration \( t \in [0,1) \) and \( \beta(t) \) is a phase pulse.

4) Proposed pulse parametrization: The remaining degree of freedom which we exploit to set the signal correlation is a phase pulse shape. We introduce a simple shaping form obtained as a nonlinear parametrization of Raised Cosine (RC) pulse which we denote as a Scaled RC (SRC) pulse. The proposed parametric SRC phase pulse is

\[
\beta(t, p) = \frac{1}{2} \left( t - p \frac{\sin 2\pi t}{2\pi} \right),
\]

where \( p \) is a real parameter. The phase pulse correspond to REC pulse for \( p = 0 \) and to RC pulse for \( p = 1 \), see Fig. 9. This parametrization has number of advantages, it does not influence the number of modulator states/signal alphabet cardinality and it has known analytical formula for bandwidth [23] (roughly the higher \( p \) the wider bandwidth).

5) Design of Binary UMP-CPM:

Lemma 17. Binary full-response CPM with \( \kappa = 1/2 \) and parametric SRC pulse (34) with \( p \simeq 2.35 \) is UMP.

Proof: Let us consider a binary case, the positive-sign alphabet is

\[
\mathcal{A}^+ = \{ s_0(t), s_1(t) \} = \left\{ e^{j\pi (\frac{t}{2} + \beta(p))}, e^{j\pi (\frac{t}{2} - \beta(p))} \right\},
\]

The correlation coefficient \( \rho = \langle s_0(t), s_1(t) \rangle = \int_0^1 s_0(t)s_1^*(t)dt \) has an analytic expression in the case of SRC pulse. The expression consists of generalized hyper-geometric functions with a zero real part, see |\( \rho \) in Fig. 10. We conclude that \( p \simeq 2.35 \) leads to the ortonormal signals and the lemma is true.

Remark 18. The proposed pulse parametrization has an extra advantage that the squared norm of the signal difference of binary alphabet is always 2 for any \( p \). The reason is simply given by zero real part of \( \rho \) for any \( p \), as has been mentioned in the proof above, then \( ||s_0(t) - s_1(t)||^2 = 2(1 - \Re\{\rho\}) = 2 \).