greater than 1 for roughly QFSK and XOR XC. We conclude that constant distance of QFSK are depicted in Fig. 7 by thin light blue color. In the same restrict on either b ≥ 1 or limiting value 1 if b ≥ 1. In Fig. 6, we plot the constant b for all indices c_A ≠ c_A', c_B ≠ c_B', c_A + c_B ≠ c_A' + c_B' and ∀c ∈ C, |c| ≤ 1. The following steps further adjust (28) to a suitable 2nd order polynomial form

\[ |\alpha|^2 \left( \|s_{c_A} - s_{c_A'}\|^2 - \delta^2 \right) - 2|\alpha| \left( \langle s_{c_A} - s_{c_A'}, s_{c_B} - s_{c_B'} \rangle \right) + \|s_{c_A} - s_{c_A'}\|^2 \geq 0, \]

(29)

\[ |\alpha|^2 - 2|\alpha| \left( \frac{\|s_{c_A} - s_{c_A'}\|^2}{\|s_{c_B} - s_{c_B'}\|^2} \right) + \frac{\|s_{c_A} - s_{c_A'}\|^2}{\|s_{c_B} - s_{c_B'}\|^2} - \delta^2 \geq 0, \]

(30)

\[ (|\alpha| - b)^2 + c \geq 0, \]

(31)

where auxiliary constants

\[ b = \frac{\langle s_{c_A} - s_{c_A'}, s_{c_B} - s_{c_B'} \rangle}{\|s_{c_B} - s_{c_B'}\|^2 - \delta^2}, \quad b \geq 0 \]

(32)

and c = b^2 + \|s_{c_A} - s_{c_A'}\|^2/\|s_{c_B} - s_{c_B'}\|^2 - \delta^2 are not functions of |\alpha|. Thus, the condition (28) has a critical |\alpha| which equals to either b if b ≤ 1 or limiting value 1 if b ≥ 1. In Fig. 6, we plot the constant b for all indices c_A ≠ c_A', c_B ≠ c_B', c_A + c_B ≠ c_A' + c_B' for QFSK and XOR XC. We conclude that constant b is always greater than 1 for roughly \( \kappa \geq 0.3 \). For practical purposes, we restrict on \( \kappa \geq 1/2 \) because than the minimal distance \( \delta^2 \) is reasonably high. The restriction implies that constant \( b \geq 1 \) and so the critical |\alpha| = 1. L.h.s of (28) for critical |\alpha| = 1 are depicted in Fig. 7 by thin light blue color. In the same figure, we chart their minimum (thick blue) and the minimal distance of QFSK \( \delta^2 \) (thick green). The lowest modulation index leading to UMP-QFSK is \( \kappa = 5/6 \).

Figure 7: Distances of hierarchical symbols corresponding to different XC symbols for QFSK and the critical parameter value \( \alpha = e^{j\psi} \). Minimal value of modulation index fulfilling the UMP condition is \( \kappa = 5/6 \) (green thick line meets blue thick line).

**Proof:** We have seen in Sec. IV-E that the condition implying UMP property (23) is

\[ \|s_{c_A} - s_{c_A'}\|^2 + |\alpha|^2 \|s_{c_B} - s_{c_B'}\|^2 - 2|\alpha| \langle s_{c_A} - s_{c_A'}, s_{c_B} - s_{c_B'} \rangle \geq |\alpha|^2 \delta^2, \]

(28)

for \( c_A \neq c_A', c_B \neq c_B', c_A + c_B \neq c_A' + c_B' \) and ∀\( \alpha \in C, |\alpha| \leq 1 \). The restriction implies that constant \( b \geq 1 \) and so the critical |\alpha| = 1. L.h.s of (28) for critical |\alpha| = 1 are depicted in Fig. 7 by thin light blue color. In the same figure, we chart their minimum (thick blue) and the minimal distance of QFSK \( \delta^2 \) (thick green). The lowest modulation index leading to UMP-QFSK is \( \kappa = 5/6 \).