QFSK with $\kappa = 1$ and XOR (b) is robust to parametrization – is uniformly most powerful (UMP). QFSK with $\kappa = 1$ and modulo sum XC (c) is not UMP due to the poor performance for the parameter $-1$.

B. Catastrophic Parameters and Paper Motivation

In the preceding section, we have seen that the HDF-MAC stage with CSIR has parametric minimal distance (asymptotic performance). For some modulation alphabets and exclusive codes, there exist such non-zero parameters (called catastrophic) which yields even zero minimal distance. This problem is well demonstrated e.g. by QPSK and complex-\textit{catastrophic} which yields even zero minimal distance. This means that one of the channel is relatively zero. Zero distance for the parametrization. Zero distance for\textit{catastrophic} parameters and therefore it is robust to the same point (equivalently, it indicates zero minimal distance parabolically dependent on $|\alpha|$ as

$$d_{\text{min}}^2(\alpha) = 2|\alpha|^2. \tag{10}$$

It has no catastrophic parameters and therefore it is robust to the parametrization. Zero distance for $\alpha = 0$ is expected because it means that one of the channel is relatively zero. This paper focuses on the design of alphabets and XCs like this. We demonstrate by Fig. 4c with QFSK $\kappa = 1$ and modulo sum XC (4), that not only modulation alphabet, but also exclusive code influence the parameter robustness.

Before we state the core idea of UMP alphabet, let us precisely define catastrophic parameters and state a couple of important lemmas based on them.

\textbf{Definition 2} (Catastrophic parameters). A catastrophic parameter is such a non-zero parameter $\alpha_{\text{cat}}$ that forces two hierarchical signals corresponding to different XC symbols to the same point (equivalently, it indicates zero minimal distance). In our notation, for some $\alpha_{\text{cat}} \neq 0$ there exists $c_A \oplus c_B \neq c_A' \oplus c_B'$ that

$$\|u_{c_A'c_B'}(\alpha_{\text{cat}}) - u_{c_A'c_B'}(\alpha_{\text{cat}}')\|^2 = 0. \tag{11}$$

C. Exclusive Code Not Implying Catastrophic Parameters

This section shows that XC must fulfill certain conditions not to imply catastrophic parameters regardless of the modulation alphabet. This reduces the number of XCs, see Tab I, involved in search for alphabets and XCs robust to the parametrization. The conditions are derived again in order to avoid catastrophic parameters.

\textbf{Theorem 3}. A matrix of XC with different symbols on the main diagonal implies $\alpha_{\text{cat}} = -1$ and XC matrix which is not symmetric over the main diagonal has $\alpha_{\text{cat}} = 1$ regardless of modulation.

\textbf{Proof}: Let two hierarchical signals correspond to the XC matrix main diagonal, $c_A = c_B, c_A' = c_B'$ i.e. $u_{c_A'c_B'}, u_{c_B'c_A}, c_A \neq c_A'$ and their XC symbols are different $c_A \oplus c_A \neq c_A' \oplus c_A'$. Equation (11) is then

$$|s_{c_A} - s_{c_A'} + \alpha(s_{c_A} - s_{c_A'})|^2 = |1 + \alpha|^2|s_{c_A} - s_{c_A'}|^2 \tag{12}$$

and $\alpha_{\text{cat}} = -1$. Similarly for non-symmetric XC matrix. Assume $u_{c_A'c_B}, u_{c_B'c_A}$ with $c_A \oplus c_B \neq c_B \oplus c_A$, equation (11) is

$$|s_{c_A} - s_{c_B} + \alpha(s_{c_B} - s_{c_A})|^2 = |1 - \alpha|^2|s_{c_A} - s_{c_B}|^2 \tag{13}$$

and $\alpha_{\text{cat}} = 1$. We conclude that XC matrix should be symmetric with the same code symbols on its main diagonal.

\textbf{Remark 4} (Suitability of bit-wise XOR XC). XOR fulfills these conditions and it is the only solution for binary and even quaternary alphabet (unfortunately it is not the only choice for e.g. octal alphabet) [17]. Once we fix XC (at least for binary and quaternary case), the only thing that influences the parameter robustness is the modulation alphabet. Therefore, from now on, we assume $\oplus$ is XOR for all cases and we relate the parametrization robustness only with particular modulation alphabets.

D. Non-Binary Linear Modulations are Catastrophic

In this section, we demonstrate that any non-binary linear modulation can never avoid catastrophic parameters, as we have seen particularly for QPSK, Fig. 4a.

\textbf{Lemma 5}. Non-binary linear modulations unavoidably have catastrophic parameters.

\textbf{Proof}: Linear modulations like QAM, PSK have dimensionality $N_s = 1$ and signals in the constellation space are