Detailed mathematical solution for the trinomial expansion of the Hardy–Weinberg principle: case of a null allele

\( r_1 \) is frequency of loss

\( p \) is frequency of \( p \) allele

\( q \) is frequency of \( q \) allele

Frequency of all possible (3) alleles: \( 1 = r_1 + p + q \)

Expansion of Hardy–Weinberg: \( 1 = (r_1 + p + q)^2 \)

\[
1 = p^2 + 2pr_1 + q^2 + 2qr_1 + r_1^2 + 2pq
\]

\#no_calls/#sampled = \( r_1^2 \)

\#Heterozygotes/#sampled = \( AB = 2pq \) \( \rightarrow pq = AB / 2 \)

\#p_homozygotes/#sampled = \( AA = p^2 + 2pr_1 = p(p + 2r_1) \)

\#q_homozygotes/#sampled = \( BB = q^2 + 2qr_1 = q(q + 2r_1) \)

Multiplying the last two lines:

\( AABB = pq(p + 2r_1)(q + 2r_1) = AB / 2(p + 2r_1)(q + 2r_1) \)

\[
2AABB / AB = (p + 2r_1)(q + 2r_1) = pq + 2pr_1 + 2qr_1 + 4r_1^2 = pq + 2r_1(p + q + r_1 + r_1) = AB / 2 + 2r_1(1 + r_1) = AB / 2 + 2r_1 + 2r_1^2
\]

\( 0 = r_1^2 + r_1 + AB / 4 - AABB / AB \)

\( r_1 = \{-1 + [1 - 4(AB / 4 - AABB / AB)]^{0.5}\} / 2 = \{[1 - AB + 4AABB / AB]^{0.5} - 1\} / 2 = [0.25 - 0.25AB + AABB / AB]^{0.5} - 0.5 \)
Detailed mathematical solution for the trinomial expansion of the Hardy–Weinberg principle: case of an extra (third) allele

\( r_g \) is frequency of insertion that produces \( pq \) chromosome

\( p \) is frequency of \( p \) allele

\( q \) is frequency of \( q \) allele

Frequency of all possible (3) alleles: \( 1=p+q+r_g \)

Expansion of Hardy–Weinberg: \( 1=(p+q+r_g)^2 \)

\[
1=p^2+q^2+2pr_g+2qr_g+r_g^2+2pq
\]

\#Heterozygotes/#sampled=\( AB=2pr_g+2qr_g+r_g^2+2pq \)

\#p_homozygotes/#sampled=\( AA=p^2 \rightarrow p=AA^{0.5} \)

\#q_homozygotes/#sampled=\( BB=q^2 \rightarrow q=BB^{0.5} \)

\( AB=2AA^{0.5}r_g+2BB^{0.5}r_g+r_g^2+2AA^{0.5}BB^{0.5} \)

\( 0=r_g^2+r_g(2AA^{0.5}+2BB^{0.5})+2AA^{0.5}BB^{0.5}-AB \)

\[
r_g=\frac{-(2AA^{0.5}+2BB^{0.5})+[(2AA^{0.5}+2BB^{0.5})^2-8AA^{0.5}BB^{0.5}+4AB]^{0.5}}{2}=[AA+2AA^{0.5}BB^{0.5}+BB-2AA^{0.5}BB^{0.5}+AB]^{0.5}-AA^{0.5}-BB^{0.5}=[AA+BB+AB]^{0.5}-AA^{0.5}-BB^{0.5}
\]