Proof of Theorem 1 and Corollary 1

Proof of Theorem 1

At first, we fix the value of $\alpha_1$ and $\alpha_2$ using thresholds $c_{1,F}$ and $c_{2,F}$ as

$$\int H(F(x_0) - c_{1,F})g_0(x_0)dx_0 = \alpha_1, \; \int H(F(x_0) - c_{2,F})g_0(x_0)dx_0 = \alpha_2,$$

where $\alpha_1 < \alpha_2 (c_{2,F} < c_{1,F})$. For simplicity, we write

$$H_{F,i}(x) = H(F(x) - c_{i,F}),$$

$$H_{i,F}(x) = H(c_{i,F} - F(x)), \; i = 1, 2.$$

Then, the pAUC with FPR being between $\alpha_1$ and $\alpha_2$ has an integral formula:

$$\begin{align*}
\text{pAUC}(F, \alpha_1, \alpha_2) &= \int \int H_{F,2}(x_0)H_{1,F}(x_0)H(F(x_1) - F(x_0))g_0(x_0)g_1(x_1)dx_0dx_1 \\
&= \int \int H_{F,2}(x_0)H_{1,F}(x_0)H_{F,2}(x_1)H(F(x_1) - F(x_0))g_0(x_0)g_1(x_1)dx_0dx_1 \\
&= \int \int H_{F,2}(x_0)H_{1,F}(x_0)H_{F,2}(x_1)H_{F,1}(x_1)H(F(x_1) - F(x_0))g_0(x_0)g_1(x_1)dx_0dx_1 \\
&\quad + \int \int H_{F,2}(x_0)H_{1,F}(x_0)H_{F,2}(x_1)H_{F,1}(x_1)H(F(x_1) - F(x_0))g_0(x_0)g_1(x_1)dx_0dx_1 \\
&= \int \int H_{F,2}(x_0)H_{1,F}(x_0)H_{F,2}(x_1)H_{F,1}(x_1)H(F(x_1) - F(x_0))g_0(x_0)g_1(x_1)dx_0dx_1 \\
&\quad + \int \int H_{F,2}(x_0)H_{1,F}(x_0)g_0(x_0)dx_0 \int H_{F,1}(x_1)g_1(x_1)dx_1.
\end{align*}$$

Similarly, the approximate pAUC is given as

$$\begin{align*}
\text{pAUC}_a(F, \alpha_1, \alpha_2) &= \int \int H_{F,2}(x_0)H_{1,F}(x_0)H_{F,2}(x_1)H_{F,1}(x_1)H_a(F(x_1) - F(x_0))g_0(x_0)g_1(x_1)dx_0dx_1 \\
&\quad + \int \int H_{F,2}(x_0)H_{1,F}(x_0)g_0(x_0)dx_0 \int H_{F,1}(x_1)g_1(x_1)dx_1.
\end{align*}$$

Note that the domain of integration can be restricted to $X = \{x | g_0(x) \neq 0, g_1(x) \neq 0\}$ without loss of generality. Hence, we will prove Theorem 1 based on this restricted domain; though, we omit the notation for simplicity.
Proof. For simplicity, we define some notations:

\[ \zeta(x) = m(\Lambda(x)), \]
\[ F_{\gamma\zeta}(x) = F(x) + \gamma \zeta(x) \]
\[ c_i'_{1,F_{\gamma\zeta}} = \frac{\partial c_i,F_{\gamma\zeta}}{\partial \gamma}, \ i = 1, 2. \]

Then, the first derivative of \( \Psi(\gamma) \) with respect to \( \gamma \) is given as

\[
\frac{\partial}{\partial \gamma} \Psi(\gamma) = \int \int H'(F_{\gamma\zeta}(x_0) - c_{2,F_{\gamma\zeta}}) (\zeta(x_0) - c'_{2,F_{\gamma\zeta}}) H_{1,F_{\gamma\zeta}}(x_0) H_{F_{\gamma\zeta}x_1}^2(x_1) \\
\times H_{1,F_{\gamma\zeta}}(x_1) H_{\sigma}(F_{\gamma\zeta}(x_1) - F_{\gamma\zeta}(x_0)) g_0(x_0) g_1(x_1) dx_0 dx_1 \\
+ \int \int H_{F_{\gamma\zeta}x_1}(x_0) H'(c_{1,F_{\gamma\zeta}} - c_{2,F_{\gamma\zeta}}) (c'_{1,F_{\gamma\zeta}} - \zeta(x_0)) H_{F_{\gamma\zeta}x_1}^2(x_1) \\
\times H_{1,F_{\gamma\zeta}}(x_1) H_{\sigma}(F_{\gamma\zeta}(x_1) - F_{\gamma\zeta}(x_0)) g_0(x_0) g_1(x_1) dx_0 dx_1 \\
+ \int \int H_{F_{\gamma\zeta}x_1}(x_0) H_1,F_{\gamma\zeta}(x_1) H_{F_{\gamma\zeta}x_1}^2(x_1) \\
\times H'(c_{1,F_{\gamma\zeta}} - F_{\gamma\zeta}(x_1)) (c'_{1,F_{\gamma\zeta}} - \zeta(x_1)) H_{\sigma}(F_{\gamma\zeta}(x_1) - F_{\gamma\zeta}(x_0)) g_0(x_0) g_1(x_1) dx_0 dx_1 \\
+ \int \int H_{F_{\gamma\zeta}x_1}(x_0) H_1,F_{\gamma\zeta}(x_0) H_{F_{\gamma\zeta}x_1}^2(x_1) \\
\times H'(c_{1,F_{\gamma\zeta}} - c_{2,F_{\gamma\zeta}}) (c'_{1,F_{\gamma\zeta}} - \zeta(x_0)) H_{\sigma}(F_{\gamma\zeta}(x_1) - F_{\gamma\zeta}(x_0)) g_0(x_0) g_1(x_1) dx_0 dx_1 \\
+ \int \left\{ H'(F_{\gamma\zeta}(x_0) - c_{2,F_{\gamma\zeta}}) (\zeta(x_0) - c'_{2,F_{\gamma\zeta}}) - H_{F_{\gamma\zeta}x_1}(x_0) (\zeta(x_0) - c'_{1,F_{\gamma\zeta}}) \right\} g_0(x_0) dx_0 TPR(F_{\gamma\zeta}, c_{1,F_{\gamma\zeta}}) \\
+ \int H_{F_{\gamma\zeta}x_1}(x_0) H_1,F_{\gamma\zeta}(x_0) g_0(x_0) dx_0 TPR'(F_{\gamma\zeta}, c_{1,F_{\gamma\zeta}}),
\]

where

\[ TPR(F_{\gamma\zeta}, c_{1,F_{\gamma\zeta}}) = \int H_{F_{\gamma\zeta}x_1}(x_1) g_1(x_1) dx_1. \]
And the first derivative is rewritten such as

\[
\frac{\partial}{\partial \gamma} \Psi(\gamma) = \int H'(F_{\gamma}(x_0) - c_{2,F_{\gamma}})(\zeta(x_0) - c'_{2,F_{\gamma}})g_0(x_0)dx_0 + \int H'(c_{1,F_{\gamma}} - F_{\gamma}(x_0))(\zeta(x_0) - c_{1,F_{\gamma}})g_1(x_1)dx_1 \\
+ \int H'(F_{\gamma}(x_1) - c_{2,F_{\gamma}})(\zeta(x_1) - c'_{2,F_{\gamma}})g_1(x_1)dx_1 + \int H'(c_{1,F_{\gamma}} - F_{\gamma}(x_1))(\zeta(x_1))g_1(x_1)dx_1 \\
+ \int \left\{ H'(F_{\gamma}(x_0) - c_{2,F_{\gamma}})(\zeta(x_0) - c'_{2,F_{\gamma}}) - H'(F_{\gamma}(x_0) - c_{1,F_{\gamma}})(\zeta(x_0) - c'_{1,F_{\gamma}}) \right\}g_0(x_0)dx_0TPR(F_{\gamma}, c_{1,F_{\gamma}}) \\
+ \int H_{F_{\gamma},2}(x_0)H_{1,F_{\gamma}}(x_0)g_0(x_0)dx_0TPR'(F_{\gamma}, c_{1,F_{\gamma}}).
\]

Since \( F_{\gamma}, c_{1,F_{\gamma}} \) and \( F_{\gamma}, c_{2,F_{\gamma}} \) are fixed, we have

\[
\frac{\partial}{\partial \gamma} \Psi(\gamma) = \text{TPR}'(F_{\gamma}, c_{2,F_{\gamma}}) \int H_{F_{\gamma},2}(x_0)H_{1,F_{\gamma}}(x_0)H_{\sigma}(F_{\gamma}(x_0) - c_{2,F_{\gamma}})g_0(x_0)dx_0 \\
+ \text{TPR}'(F_{\gamma}, c_{1,F_{\gamma}}) \int H_{F_{\gamma},2}(x_0)H_{1,F_{\gamma}}(x_0)H_{\sigma}(c_{1,F_{\gamma}} - F_{\gamma}(x_0))g_0(x_0)dx_0 \\
+ \int \text{TPR}'(F_{\gamma}, c_{1,F_{\gamma}}) \int H_{F_{\gamma},2}(x_0)H_{1,F_{\gamma}}(x_0)H_{\sigma}(c_{1,F_{\gamma}} - F_{\gamma}(x_0))g_0(x_0)dx_0 \\
\times H_{1,F_{\gamma}}(x_1)H_{\sigma}(F_{\gamma}(x_1) - F_{\gamma}(x_0))H_{\sigma}(\zeta(x_1) - \zeta(x_0))g_0(x_0)g_1(x_1)dx_0dx_1.
\] (B.1)

Next, we investigate the behavior of \( \text{TPR}'(F_{\gamma}, c_{2,F_{\gamma}}) \). The value of \( F_{\gamma}, c_{2,F_{\gamma}} \) is fixed, so we have

\[
\text{FPR}'(F_{\gamma}, c_{2,F_{\gamma}}) = \int H'(F_{\gamma}(x_0) - c_{2,F_{\gamma}})(\zeta(x_0) - c'_{2,F_{\gamma}})g_0(x_0)dx_0 = 0.
\]

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Then we have

\[ c'_{2,F_{\zeta}} = \frac{\int H'(F_{\zeta \gamma}(x_0) - c_{2,F_{\zeta}}) \zeta(x_0) g_0(x_0) dx_0}{\int H'(F_{\zeta \gamma}(x_0) - c_{2,F_{\zeta}}) g_0(x_0) dx_0}, \]

where the denominator is not zero because the domain of integration is \( \mathcal{X} = \{x | g_0(x) \neq 0, g_1(x) \neq 0\} \). By substituting it into \( \text{TPR}'(F_{\zeta \gamma}, c_{2,F_{\zeta}}) \), we have

\[ \text{TPR}'(F_{\zeta \gamma}, c_{2,F_{\zeta}}) = \int \int K(x_0, x_1) \zeta(x_1) g_0(x_0) g_1(x_1) dx_0 dx_1 - \int \int K(x_0, x_1) \zeta(x_0) g_0(x_0) g_1(x_1) dx_0 dx_1, \]

(B.2)

where

\[ K(x_0, x_1) = H'(F_{\zeta \gamma}(x_0) - c_{2,F_{\zeta}}) H'(F_{\zeta \gamma}(x_1) - c_{2,F_{\zeta}}) \]

Then, the numerator becomes

\[ \int \int K(x_0, x_1) \left( \zeta(x_1) - \zeta(x_0) \right) g_0(x_0) g_1(x_1) dx_0 dx_1 = \int \int K(x_0, x_1) \left( \zeta(x_0) - \zeta(x_1) \right) g_0(x_1) g_0(x_0) dx_0 dx_1 \]

\[ = \frac{1}{2} \int \int K(x_0, x_1) \left( \zeta(x_1) - \zeta(x_0) \right) \left( g_0(x_0) g_1(x_1) - g_0(x_1) g_0(x_0) \right) dx_0 dx_1 \]

\[ = \frac{1}{2} \int \int K(x_0, x_1) \left( \zeta(x_1) - \zeta(x_0) \right) \left( \Lambda(x_1) - \Lambda(x_0) \right) g_0(x_0) g_0(x_1) dx_0 dx_1 > 0. \]  

(B.3)

Hence, we have

\[ \text{TPR}'(F_{\zeta \gamma}, c_{i,\gamma}) \geq 0, \ i = 1, 2, \]

because we can replace \( c_{2,F_{\zeta}} \) with \( c_{1,F_{\zeta}} \), and have the same result.

By looking at the third term in Equation (B.1), we find

\[ H_{F_{\zeta \gamma}(x_0) H_{1,F_{\zeta}}(x_0) H_{F_{\zeta \gamma}(x_1) H_{1,F_{\zeta}}(x_1)v'(F_{\zeta \gamma}(x_1) - F_{\zeta \gamma}(x_0))} \]

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is invariant to the exchange of $x_0$ for $x_1$ like $K(x_0, x_1)$. Hence by the same argument above, we have

$$
\int \int H_{F, \gamma, 2}(x_0)H_{1, F, \gamma}(x_0)H_{F, \gamma, 2}(x_1) \\
\times H_{1, F, \gamma}(x_1)H'_{\gamma}(F_{\gamma}(x_1) - F_{\gamma}(x_0))(\zeta(x_1) - \zeta(x_0))g_0(x_0)g_1(x_1)dx_0dx_1
$$

$$
= \frac{1}{2} \int \int H_{F, \gamma, 2}(x_0)H_{1, F, \gamma}(x_0)H_{F, \gamma, 2}(x_1)H_{1, F, \gamma}(x_1)H'_{\gamma}(F_{\gamma}(x_1) - F_{\gamma}(x_0))(\zeta(x_1) - \zeta(x_0)) \\
\times (\Lambda(x_1) - \Lambda(x_0))g_0(x_0)g_0(x_1)dx_0dx_1
$$

$$
> 0.
$$

As a result, we have

$$
\frac{\partial}{\partial \gamma} \Psi(\gamma) > 0.
$$

Finally, we have

$$
p_{\text{AUC}}(F; \alpha_1, \alpha_2) < \lim_{\gamma \to \infty} \Psi(\gamma)
$$

$$
= \lim_{\gamma \to \infty} \text{pAUC}_{\sigma}\left[ \gamma \left\{ \frac{F}{\gamma} + \zeta \right\}, \alpha_1, \alpha_2 \right]
$$

$$
= \lim_{\gamma \to \infty} \text{pAUC}_{\sigma}\left( \frac{F}{\gamma} + \zeta, \alpha_1, \alpha_2 \right)
$$

$$
= \text{pAUC}(\zeta, \alpha_1, \alpha_2)
$$

$$
= \text{pAUC}(\Lambda, \alpha_1, \alpha_2).
$$

Since the inequation above holds for any $F$, $\lim_{\gamma \to \infty} \Psi(\gamma)$ is an upper bound of $p_{\text{AUC}}(F; \alpha_1, \alpha_2)$. On the other hand, from the definition of the supremum of $p_{\text{AUC}}(F; \alpha_1, \alpha_2)$ we have

$$
\Psi(\gamma) = \text{pAUC}_{\sigma}\left( F + \gamma \cdot m(\Lambda), \alpha_1, \alpha_2 \right) \leq \sup_{F} \text{pAUC}_{\sigma}(F; \alpha_1, \alpha_2).
$$

By taking the limit of $\Psi(\gamma)$ as $\gamma$ approaches $\infty$, we have

$$
\lim_{\gamma \to \infty} \Psi(\gamma) \leq \sup_{F} \text{pAUC}_{\sigma}(F, \alpha_1, \alpha_2).
$$

As a result, we have

$$
\lim_{\gamma \to \infty} \Psi(\gamma) = \sup_{F} \text{pAUC}_{\sigma}(F, \alpha_1, \alpha_2),
$$

which proves Theorem 1.
Proof of Corollary 1

At first we fix the value of FPR as \( \text{FPR}(F_{\gamma}, c_{F_{\gamma}}) = \alpha \). Then, the first derivative of \( \text{TPR}(F_{\gamma}, c_{F_{\gamma}}) \) regarding to \( \gamma \) is given from (B.2) and (B.3) as

\[
\begin{align*}
\text{TPR}'(F_{\gamma}, c_{F_{\gamma}}) &= \frac{1}{2} \int \int K^*(x_0, x_1) \left( \eta(x_1) - \eta(x_0) \right) \left( \Lambda(x_1) - \Lambda(x_0) \right) g_0(x_0) g_0(x_1) dx_0 dx_1 \\
&\quad \div \int H'(F_{\gamma}(x_0) - c_{F_{\gamma}}) g_0(x_0) dx_0,
\end{align*}
\]

where

\[
K^*(x_0, x_1) = H'(F_{\gamma}(x_0) - c_{F_{\gamma}}) H'(F_{\gamma}(x_1) - c_{F_{\gamma}}),
\]

and the denominator is not zero because the domain of integration is \( X = \{ x | g_0(x) \neq 0, g_1(x) \neq 0 \} \). The domain of the integration in the numerator is determined by \( K^* \), which is dependent on an arbitrary score function \( F \). Hence for any \( F \), \( \text{TPR}'(F_{\gamma}, c_{F_{\gamma}}) \geq 0 \) only if \( \eta = m(\Lambda) \), where \( m \) is a strictly increasing function. The sufficiency is confirmed easily.