**Online Resources.** The following sections are Online Resources for the paper


### Online Resource A  Block-diagonal matrices

Equations (5) – (9) describe the creation of several block-diagonal matrices from the projection and aging matrices $A$, $U$, $F$, $D_U$, and $D_F$. These all have a similar structure, and only $A$ was shown in the paper. The complete set of block-diagonal matrices is:

$$
A = \begin{pmatrix}
A_1 & & \\
& \ddots & \\
& & A_\omega
\end{pmatrix}
$$

(1)

$$
U = \begin{pmatrix}
U_1 & & \\
& \ddots & \\
& & U_\omega
\end{pmatrix}
$$

(2)

$$
F = \begin{pmatrix}
F_1 & & \\
& \ddots & \\
& & F_\omega
\end{pmatrix}
$$

(3)

$$
D_U = \begin{pmatrix}
D_{U_1} & & \\
& \ddots & \\
& & D_{U_\omega}
\end{pmatrix}
$$

(4)

$$
D_F = \begin{pmatrix}
D_{F_1} & & \\
& \ddots & \\
& & D_{F_\omega}
\end{pmatrix}
$$

(5)

### Online Resource B  A brief survey of matrix calculus

Matrix calculus permits the consistent differentiation of scalar-, vector-, and matrix-valued functions of scalar, vector, or matrix arguments. This appendix presents a brief statement of some essential results. More detail can be found in Caswell (2007, 2009, 2010, Klepac and Caswell 2010). A good introductory treatment is found in Abadir and Magnus (2005), and the most detailed presentation is the book of Magnus and Neudecker (1988).

There exist several conventions for matrix calculus, differing in their arrangements of the matrix and vector entries. The best is that of Magnus and Neudecker (1985, 1988); it is called the vector-arrangement method in the review paper of Nel (1980).

If $x$ and $y$ are scalars, the derivative of $y$ with respect to $x$ is the familiar derivative $dy/dx$. If $y$ is a $n \times 1$ vector and $x$ a scalar, the derivative of $y$ with respect to $x$ is the $n \times 1$ vector

$$
\frac{dy}{dx} = \begin{pmatrix}
\frac{dy_1}{dx} \\
\vdots \\
\frac{dy_n}{dx}
\end{pmatrix}
$$

(6)

If $y$ is a scalar and $x$ a $m \times 1$ vector, the derivative of $y$ with respect to $x$ is the $1 \times m$ gradient vector

$$
\frac{dy}{dx} = \begin{pmatrix}
\frac{\partial y}{\partial x_1} & \cdots & \frac{\partial y}{\partial x_m}
\end{pmatrix}
$$

(7)

Note the orientation of $dy/dx$ as a column vector and $dy/dx^T$ as a row vector.

If $y$ is a $n \times 1$ vector and $x$ a $m \times 1$ vector, the derivative of $y$ with respect to $x$ is the $n \times m$ Jacobian matrix

$$
\frac{dy}{dx} = \begin{pmatrix}
\frac{dy_1}{dx_1} & \cdots & \frac{dy_1}{dx_m} \\
\vdots & \ddots & \vdots \\
\frac{dy_n}{dx_1} & \cdots & \frac{dy_n}{dx_m}
\end{pmatrix}
$$

(8)

Derivatives involving matrices are written by transforming the matrices into vectors using the vec operator (which stacks the columns of the matrix into a column vector), and then applying the rules for vector differentiation. Thus, the derivative of the $m \times n$ matrix $Y$ with respect to the $p \times q$ matrix $X$ is the $mn \times pq$ matrix

$$
\begin{pmatrix}
\vec{Y} \\
\vec{X}
\end{pmatrix}
$$

(9)

For notational convenience, I will write $\vec{Y}^T$ for $(\vec{Y})^T$.

These definitions (unlike some alternatives; see Magnus and Neudecker 1985) lead to the familiar chain rule. If $Y$ is a function of $X$ and $X$ is a function of $Z$, then

$$
\frac{d\vec{Y}}{d\vec{Z}} = \frac{d\vec{Y}}{d\vec{X}} \frac{d\vec{X}}{d\vec{Z}}
$$

(10)

The derivatives of matrices are constructed by forming the differentials of the expressions involving the matrices. The differential of a matrix (or vector) is the matrix (or vector) of differentials of the elements; i.e.,

$$
\frac{dX}{dx_{ij}}
$$

(11)
If, for vectors \( x \) and \( y \) and some matrix \( Q \), it can be shown that
\[
dy = Qdx
\]  \hfill (12)

then
\[
\frac{dy}{dx^T} = Q.
\]  \hfill (13)

(the “first identification theorem” of Magnus and Neudecker (1985)).

The combination of the chain rule and the identification theorem permits more complicated expressions involving differentials to be turned into derivatives with respect to an arbitrary vector, say \( u \). If
\[
dy = Qdx + Rdz
\]  \hfill (14)

then
\[
\frac{dy}{du^T} = Q\frac{dx}{du^T} + R\frac{dz}{du^T}
\]  \hfill (15)

for any \( u \).

We will make extensive use the Kronecker product, defined as
\[
A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots \\ a_{21}B & a_{22}B & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}
\]  \hfill (16)

The vec operator and the Kronecker product are related (Roth 1934); if
\[
Y = ABC
\]  \hfill (17)

then
\[
\text{vec } Y = (C^T \otimes A) \text{ vec } B.
\]  \hfill (18)

Online Resource C  Additional figures

This section presents figures showing the sensitivity and elasticity of population growth rate \( \lambda \) to changes in age-stage specific survival, for eight populations of Scotch broom from the paper of Parker (2000).
Fig. 1: Sensitivity of $\lambda$ to stage-specific survival as a function of age, for 8 populations of Scotch broom (\textit{Cytisus scoparius}) using data from Parker (2000). Stages: 1 = seeds, 2 = seedlings, 3 = juveniles, 4 = small adults, 5 = medium adults, 6 = large adults, 7 = extra-large adults.
Fig. 1: (cont’d.) Sensitivity of $\lambda$ to stage-specific survival as a function of age, for 8 populations of Scotch broom. Stages defined as in Figure 1.
Fig. 2: Elasticity of $\lambda$ to stage-specific survival as a function of age, for 8 populations of Scotch broom. Stages defined as in Figure 1.
Fig. 2: (cont’d.) Elasticity of $\lambda$ to stage-specific survival as a function of age, for 8 populations of Scotch broom. Stages as in Figure 1.