Electronic Supplementary Material for:
Testing for Poisson Arrivals in INAR(1) Processes

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S1 Cumulants

**Lemma 1** Let \((Y_1, \ldots, Y_r)\) denote a \(r\)-variate non-negative integer-valued random variable with \(\mathbb{E}(Y_i^r) < \infty\) for all \(i \in \{1, \ldots, r\}\).

(i) If \(Y_i = Y\) for all \(i \in \{1, \ldots, k\}\), then \(\text{cum}(Y, \ldots, Y) = \kappa_k(Y)\).

(ii) If any group of \(Y\)'s is independent of the remaining \(Y\)'s, then \(\text{cum}(Y_1, \ldots, Y_r) = 0\).

(iii) The cumulant function is additive, i.e. \(\text{cum}(Y_1 + Z_1, \ldots, Y_r) = \text{cum}(Y_1, \ldots, Y_r) + \text{cum}(Z_1, \ldots, Y_r)\).

(iv) \(\text{cum}(a_1Y_1, \ldots, a_rY_r) = a_1 \cdots a_r \text{cum}(Y_1, \ldots, Y_r)\) for \(a_1, \ldots, a_r\) constant.

(v) If \(r \geq 2\), then the \(r\)th joint cumulant is shift-invariant, i.e. it holds that \(\text{cum}(Y_1 + c_1, \ldots, Y_r + c_r) = \text{cum}(Y_1, \ldots, Y_r)\), where \(c_1, \ldots, c_r \in \mathbb{R}\) constant.

(vi) For any \(r \in \mathbb{N}\),

\[
\mathbb{E}[Y_1 \cdots Y_r] = \sum_{\pi \in \Pi_r} \prod_{B \in \pi} \text{cum}(Y_i : i \in B),
\]

where \(\Pi_r\) and \(B\) are as in (6).

(vii) Let \(X_{1,i}, X_{2,i}, \ldots, X_{r,i}\) be sequences of non-negative random variables, such that \(X_{1,i}\) and \(X_{k,l}\) are independent for \(j \neq l\) for all \(i, k \in \{1, \ldots, r\}\) and

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such that \( E[(\sum_{i=0}^{\infty} X_{j,i})^r] < \infty \) for \( j \in \{1, \ldots, r\} \). Then
\[
\text{cum} \left( \sum_{i=0}^{\infty} X_{1,i}, \sum_{i=0}^{\infty} X_{2,i}, \ldots, \sum_{i=0}^{\infty} X_{r,i} \right) = \sum_{i=0}^{\infty} \text{cum} (X_{1,i}, X_{2,i}, \ldots, X_{r,i}).
\]

Proof For (i) through (v), we refer to Theorem 2.3.1 in [Brillinger 1981], relation (vi) is well-known. (vii): As the condition \( E[(\sum_{i=0}^{\infty} X_{j,i})^r] < \infty \) ensures that the expression is well-defined ([Brillinger 1981] Definition 2.3.1), we prove this statement via application of the defining equation (6) of joint cumulants. In order to allow for the changing of the order of integration (i.e., taking the mean) and summation, we apply Lesbesgue’s monotone convergence theorem. Since the \( X_{ji} \)’s are non-negative and since the arising expectations have an upper bound in \( \max_{1 \leq j \leq r} \{ E[(\sum_{i=0}^{\infty} X_{ji})^r] \} < \infty \), this theorem is applicable. Using these argumentations, we first find with (iii) that
\[
\text{cum} \left( \sum_{i_1=0}^{\infty} X_{1,i_1}, \sum_{i_2=0}^{\infty} X_{2,i_2}, \ldots, \sum_{i_r=0}^{\infty} X_{r,i_r} \right) = \ldots = \sum_{i_1=0}^{\infty} \cdots \sum_{i_r=0}^{\infty} \text{cum} (X_{1,i_1}, X_{2,i_2}, \ldots, X_{r,i_r}),
\]
which, using (ii), concludes the proof.

S2 Further Proofs

S2.1 Proof of Theorem 2.1

In the sequel, we use the notion of the \( i \)-time iteration of the \( \circ \) operator, i.e., the application of \( i \) independent thinning operations on a random variable \( X \). We define
\[
\alpha^i \circ X := \underbrace{\alpha \circ \alpha \circ \cdots \circ \alpha}_{i \text{ times}} \circ X = \sum_{j=1}^{X} \prod_{k=1}^{i} \xi_{j,k}, \tag{S2.1}
\]
where \( \xi_{a,b} \) for \( (a,b) \in \mathbb{N} \times \{1, \ldots, i\} \) are i.i.d. Bernoulli rvs, independent of \( X \), with probability of success \( \alpha \). We set \( \alpha^0 \circ X := X \). Then the following distributional results for INAR(1) processes holds ([Al-Osh and Alzaid 1987] eq. (2.2)):
\[
Y_t \overset{D}{=} \sum_{j=0}^{\infty} \alpha^j \circ \epsilon_{t-j} \quad \text{and} \quad (Y_k, Y_l) \overset{D}{=} \left( Y_k, \alpha^{l-k} \circ Y_k + \sum_{s=0}^{l-k-1} \alpha^s \circ \epsilon_{t-s} \right), \tag{S2.2}
\]
for any \( l, k \in \mathbb{Z}, l \geq k \). Note that in the cited reference, the rv \( \alpha^i \circ X \) does not represent iterated thinning as we defined it above, but one single thinning with parameter \( \alpha^i \). As the resultant rvs are equal in distribution, however, the result holds.
The joint cumulant is well-defined as the marginal distribution of a Poisson INAR(1) process is Poisson distributed and thus all moments are finite. Let \( j_0 := 0 \leq j_1 \leq j_2 \leq \ldots \leq j_r \) and \( j_r \in \mathbb{N}_0 \). In the following, we use the law of total cumulence, introduced in Brillinger (1969). Furthermore, we use the notation of Lemma 1 (vi) and (S2.1) to obtain

\[
\text{cum} \left( \alpha^{j_1} \circ \epsilon_0, \alpha^{j_2} \circ \epsilon_0, \ldots, \alpha^{j_r} \circ \epsilon_0 \right) = \sum_{\pi \in \Pi_{r}} \text{cum} \left( \text{cum} \left( \alpha^{j_1} \circ \epsilon_0 : l \in B \right) : B \in \pi \right)
\]

Now, since equality in distribution implies equality of the corresponding cumulants, we apply (S2.2) and Lemma 1 (vii) repeatedly (note the mutual independence of the \( \epsilon \)'s and the independence of \( \epsilon_t \) of \( Y_s \) for \( s < t \)). We obtain

\[
\text{cum} \left( Y_{i_1}, \ldots, Y_{i_s} \right) = \sum_{s=0}^{\infty} \text{cum} \left( \alpha^{j_1-s} \circ \epsilon_{j_1-s}, \ldots, \alpha^{j_r-s} \circ \epsilon_{j_r-s} \right)
\]

and conclude the proof with an appeal to (S2.3).

S2.2 Proof of Lemma 3.2

Note that the expressions in \( \sigma_t^2 \) consist of terms of the form \( \mathbb{E}[Y_0 Y_a Y_1 Y_t Y_{t+1}] \) with \( a \in \{0,1\} \) and \( b \in \{t, t+1\} \) due to stationarity and \( t \in \mathbb{N}_0 \). For each
term, we apply Lemma [iv] to convert the expectation into an expression of the cumulants, finding
\[ \mathbb{E}[Y_0 Y_1 Y_2 Y_3 Y_{t+1}] = \sum_{\pi \in \mathcal{P}_n} \prod_{B \in \pi} \text{cum}(Y_i; i \in B). \]  
(S2.4)

Let \( t > 0 \), and define the summands of the latter expression as \( \nu_{\tau}(a, b) \) for each \( \pi \in \mathcal{P}_n \). We calculate \( \nu_t(0, t) + \nu_t(1, t+1) - \nu_t(0, t+1) - \nu_t(1, t) =: \nu_t \) by using Theorem 2.1 i.e., we evaluate this expression for different forms of partitions \( \pi \) separately. We denote each partition in an obvious manner. For instance, the notation \((123)(456)\) stands for the partition consisting of two blocks, where the three random variables with the three lowest indices, represented by \(1, 2 \) and \(3\), are in one block, and the other random variables are contained in the second block. So, for \( a = 0 \) and \( b = t \), this corresponds to the summand \( \nu_t(123)(456)(0, t) = \text{cum}(Y_0, Y_0, Y_1) \cdot \text{cum}(Y_t, Y_t, Y_{t+1}) \).

Note that \( a \) and \( b \) correspond to the second and fifth index, all other indices are equal. Hence, for any partition \( \pi \in \mathcal{P}_n \), for which none of either the highest or the lowest indices of any block is either 2 or 5, it holds that \( \nu_{\tau} = 0 \), for example, \( \nu_t(123456) = 0 \). Furthermore, if only one of the highest or lowest index of any block of a given partition \( \pi \in \mathcal{P}_n \) is either 2 or 5, it also holds that \( \nu_{\tau} = 0 \). For example, \( \nu_t(14)(2356) = \mu_2^2 \alpha^t + \alpha^t - \alpha^{t+1} - \alpha^t = 0 \).

Thus, it suffices to consider only those partitions \( \pi \in \mathcal{P}_n \), in which both 2 and 5 appear as either the highest or the lowest index of any block, given that these blocks contain at least two elements. In what follows, these partitions are referred to as the “remaining partitions”. We list the calculation for all possible partitions separately in ascending order with respect to the number of blocks in the partitions. When using letters instead of numbers, we want to express that a certain relation holds for all possible permutations of a given partition, e.g., \((a)(bcdef)\) stands for the partitions \((1)(23456), (2)(13456), (3)(12456)\) and so forth.

We have already shown that \( \nu_t(123456) = 0 \). It similarly follows that \( \sum_{\pi = (a)(bcdef)} \nu_{\pi} = 0 \). For partitions \((ab)(cdef)\), the remaining partitions are \((15)(2346), (16)(2345), (25)(1346), (26)(1345)\). By Theorem 2.1, we find \( \nu_{\tau}(a, b) = \mu_2^2 \alpha^{t+1+b-a} \) for each of these partitions. For the case \((abc)(def)\), the remaining partitions are \((135)(246), (136)(245), (145)(236)\) and \((146)(235)\). For each of these partitions, \( \nu_{\tau}(a, b) = \mu_2^2 \alpha^{t+1+b-a} \).

The only remaining partition for the case \((a)(b)(cdef)\) is \((1)(23456)(6)\). Thus, \( \sum_{\pi = (a)(bcdef)} \nu_{\pi} = -\mu_2^3 \alpha^{t-1}(1-\alpha)^2 \). For \((ab)(cd)(ef)\), there are 15 partitions to be considered. It is clear that for each partition, \( \nu_{\tau}(a, b) = \mu_2^3 \alpha^{t+1+b+t+1-a} \) with two “+” and two “−” operators. A simple count yields: the case \( \nu_{\tau}(a, b) = \mu_2^3 \alpha^{t+1+b+t+1-a} \) occurs once, the case \( \nu_{\tau}(a, b) = \mu_2^3 \alpha^{t+1+b-t-1+a} \) occurs 2 times, the case \( \nu_{\tau}(a, b) = \mu_2^3 \alpha^{t+1+b-t+1-a} \) occurs 2 times, the case \( \nu_{\tau}(a, b) = \mu_2^3 \alpha^{t+1+b-t-1-a} \) occurs 4 times, and the case \( \nu_{\tau}(a, b) = \mu_2^3 \alpha^{t+1+b+t-1-a} \) occurs 6 times.

For \((a)(bc)(def)\), we proceed similarly, the count yields: the case \( \nu_{\tau}(a, b) = \mu_2^3 \alpha^{2t-b-a+1} \) occurs once, the case \( \nu_{\tau}(a, b) = \mu_2^3 \alpha^{b+a-1} \) occurs once, the case
\[ \nu_\alpha(a, b) = \mu_3^2 \alpha^{b-a+1} \] occurs 4 times, the case \[ \nu_\nu(a, b) = \mu_4^2 \alpha^{b-a+t} \] occurs 8 times, and the case \[ \nu_\nu(a, b) = \mu_5^2 \alpha^{t+1+b-a} \] occurs 8 times.

In the case \((ab)(cd)(ef)\), we first take a closer look at the remaining partitions. Here, some partitions cancel each other out, i.e., \(\nu_{12}(12)(3)(6) + \nu_{(12)(3)(4)(56)} = \nu_{(12)(3)(4)(56) + \nu_{(12)(3)(4)(56)} = 0}\). A simple count of the remaining partitions yields: the case \(\nu_\nu(a, b) = \mu_3^2 \alpha^{b-a-1}\) occurs once, the case \(\nu_\nu(a, b) = \mu_4^2 \alpha^{b-a+2t+1}\) occurs once, the case \(\nu_\nu(a, b) = \mu_4^2 \alpha^{b-a+t} \) occurs 4 times, the case \(\nu_\nu(a, b) = \mu_5^2 \alpha^{b-a+t+1}\) occurs 2 times, and the case \(\nu_\nu(a, b) = \mu_5^2 \alpha^{b-a+t-1}\) occurs 2 times.

For \((a)(b)(c)(de)f\), the remaining partitions are \((1)(23)(4)(6)\) and \((1)(245)(3)(6)\), and in the final case, \((a)(b)(d)(c)(ef)\), the remaining partition is \((125)(3)(4)(6)\).

Now, we can conclude from (S2.4) that, for \(t > 0\),
\[
E[(Y_2^2 Y_{-1} - Y_0 Y_2^2 Y_{-1})] = - \mu_2^2 (1 - \alpha)^2 (8 \alpha^{2t} + \mu_Y [4 \alpha^{t+1} + 8 \alpha^{2t-1} + 8 \alpha^{2t} + 6 \alpha^{3t-1}]) + \mu_2^2 (4 \alpha^{t+2} + 2 \alpha^{2t-2} (1 + \alpha)^2) + \mu_3^2 \alpha^{t-1}.
\] (S2.5)

Next, let us consider the case \(t = 0\). We use the same approach as above.

Note that in this case, the ordering is altered in the sense that we consider the expectation \(E[Y_0 Y_0 Y_0 Y_1 Y_1 Y_2]\) with \(a, b \in \{0, 1\}\). We denote \(\nu_\nu(a, b)\) and \(\nu_\nu'\) in analogy to the case \(t > 0\). All except the third and fourth indices are equal.

Hence, for any partition \(\pi \in \Pi_6\), for which none of either the highest or the lowest indices of any block is either 3 or 4, it holds that \(\nu_\nu = 0\). For example, \(\nu_{(123)(456)} = 0\). Furthermore, certain pairs of partitions cancel each other out. This happens if there is a partition in which 3 is either the highest or lowest index in one block and 4 is not, and if there is the corresponding partition in which 4 is the respective highest or lowest index and 3 is not. For example, \(\nu_{(123)(456)} + \nu_{(1)(24)(35)(6)} = 0\). In what follows, we will refer to those partitions not included in one of these cases above as the “remaining partitions”, and we proceed analogously to the case \(t > 0\).

The fact that \(\nu_{(123)(456)} = 0\) and \(\nu_{(1)(23)(45)(6)} = 0\) follows analogously as before.

The remaining partitions in the case \((ab)(cde)f\) are \((12)(3456)\), \((34)(1256)\) and \((56)(1234)\). For partitions of the form \((abc)(def)\), we find \(\nu_{(123)(456)} = \mu_3^2 (2a - 2)\), \(\nu_{(124)(356)} = \mu_3^2 (2a - 2a^2)\) and \(\nu_{(125)(346)} = \nu_{(126)(345)} = \nu_{(134)(256)} = \nu_{(156)(234)} = \mu_3^2 (\alpha - \alpha^2)\).

The case \((a)(b)(cde)f\) yields the summands \(\nu_{12}(12)(3)(5) = 2 \mu_3^2 (1 - \alpha)\). For the partitions \((ab)(cd)(cde)\), we use analogous argumentation as in the case \(t > 0\). The count yields: \(\nu_\nu'(a, b) = \mu_3^2 \alpha^{b-a} \) occurs times, the case \(\nu_\nu'(a, b) = \mu_3^2 \alpha^{b-a} \) occurs 2 times, the case \(\nu_\nu'(a, b) = \mu_3^2 \alpha^{b-a} \) occurs 2 times, the case \(\nu_\nu'(a, b) = \mu_3^2 \alpha^{b-a} \) occurs 4 times, and the case \(\nu_\nu(a, b) = \mu_3^2 \alpha^{b-a} \) occurs 6 times.

For \((a)(bc)(def)\), the count of the remaining partitions yields: the case \(\nu_\nu'(a, b) = \mu_3^2 \alpha^{b-a} \) occurs 2 times, the case \(\nu_\nu'(a, b) = \mu_3^2 \alpha^{b-a} \) occurs 4 times, the case \(\nu_\nu'(a, b) = \mu_3^2 \alpha^{b-a} \) occurs 6 times.
\[ \nu'_\pi(a, b) = \mu_1^a \alpha^{b-1} \] occurs 4 times, the case \( \nu'_\pi(a, b) = \mu_1^a \alpha^b \) occurs 2 times, the case \( \nu'_\pi(a, b) = \mu_1^a \alpha^{-b+a+1} \) occurs 4 times, and the case \( \nu'_\pi(a, b) = \mu_1^a \alpha^{b-a+1} \) occurs 8 times.

For the partitions \((ab)(cd)(e)(f)\), a simple count of the remaining partitions yields: the case \( \nu'_\pi(a, b) = \mu_1^a \alpha^{b+a} \) occurs 2 times, the case \( \nu'_\pi(a, b) = \mu_1^a \alpha^{2-b-a} \) occurs 2 times, the case \( \nu'_\pi(a, b) = \mu_1^a \alpha^{-b+a} \) occurs 2 times, the case \( \nu'_\pi(a, b) = \mu_1^a \alpha^{b-a+1} \) occurs 4 times, and the case \( \nu'_\pi(a, b) = \mu_1^a \alpha^{b-a+1} \) occurs 8 times.

For the penultimate case, \((a)(b)(c)(d)e\), the remaining partitions are \((1)(2)(5)(346), (1)(2)(4)(356), (2)(5)(6)(134)\) and \((1)(5)(6)(234)\), yielding the summand \(4\mu_1^4(1-\alpha)\).

For the final case \((a)(b)(c)(d)(ef)\), the remaining partition is \((1)(34)(2)(5)(6)\) and we obtain \(\nu'(1)(34)(2)(5)(6) = 2\mu_1^2(1-\alpha)\).

Combining all of these results yields

\[
\sum_{\pi \in B_6} \nu'_\pi = 2\mu_1^2(1-\alpha) (4\alpha + \mu_Y(2 + 6\alpha + 6\alpha^2) + \mu_1^2(2 + 6\alpha + \mu_1^2)) .
\]

S2.3 Proof of Corollary 3.3

We begin by noticing that for any \(p \in [0; 1]\) and any \(N \in \mathbb{N}\) with \(N \geq 2\),

\[
\sum_{t=1}^{N-1} \left(1 - \frac{t}{N}\right) p^{t-1} = \frac{1}{1-p} - \frac{1}{N} \frac{1-p^N}{(1-p)^2} = \sum_{t=1}^{\infty} p^{t-1} - \frac{1}{N(1-p^2)} .
\] 

(S2.6)

Now, since the process \((Y_t)_{t \in \mathbb{Z}}\) is stationary and since \(\mathbb{E}[\hat{\beta}_T(1)] = 0\), it is easily seen that

\[
\text{Var} \left( \sqrt{T-1} \hat{\beta}_T(1) \right) = \mathbb{E} \left[ (Y_0^2 Y_{t-1} - Y_0 Y_{t-1}^2)^2 \right] + 2 \sum_{t=1}^{T-2} \frac{T-1-t}{T-1} \mathbb{E} \left[ (Y_0^2 Y_{t-1} - Y_0 Y_{t-1}^2)(Y_t^2 Y_{t-1} - Y_t Y_{t-1}^2) \right] .
\]
By (S2.5), \( E[(Y_0^2Y_{t-1} - Y_0Y_{t-1}^2)] \) is an expression consisting of terms with coefficients \( \alpha^{t-1}, (\alpha^2)^{t-1} \) and so forth, thus (S2.6) is applicable. We find

\[
E[(Y_0^2Y_{t-1} - Y_0Y_{t-1}^2)] + 2\sum_{t=1}^{\infty} E[(Y_0^2Y_{t-1} - Y_0Y_{t-1}^2)] + \frac{1}{T-1} \left[ 16\mu_Y^2\alpha^2 \frac{1 - \alpha^{2T-2}}{(1 + \alpha)^2} + \mu_Y^3 \left( 8\alpha^2(1 - \alpha^{T-1}) \right. \right.
\]

\[
+ 16(\alpha + \alpha^2)(1 - \alpha^{2T-2}) \left. \frac{1 - \alpha^{2T-2}}{1 + \alpha + \alpha^2} \right] + \frac{1}{T-1} \left[ 8\mu_Y^4\alpha(1 - \alpha^{T-1}) + 4(1 - \alpha^{2T-2}) + 2\mu_Y^5(1 - \alpha^{T-1}) \right],
\]

concluding the proof.

S2.4 Proof of Lemma 3.4

The relations for \( \tau_{1,1}, \tau_{1,2} \) and \( \tau_{2,2} \) were proven in Lemma A.5.1 in [Schweer and Weiß (2014)]. For the calculation of \( \tau_{1,3} \), we first consider \( E[Y_0^2 Y_k^2] = \sum_{\pi \in \Pi_4} \prod_{B \in \pi} \text{cum}(Y_i : i \in B) \), defining the summands of the latter expression as \( \rho_\pi \) for each \( \pi \in \Pi_4 \). Let the notation of the partitions be the same as that in the proof of Lemma 3.2. With Theorem 2.1, it follows that \( \rho_{(1234)} = \mu_Y \alpha^k \). Furthermore, \( \sum_{\pi=(a)(bcd)} \rho_\pi = \mu_Y^2(1 + 3\alpha^k) \) and \( \sum_{\pi=(ab)(cd)} \rho_\pi = \mu_Y^2(3\alpha^k) \). For the partitions consisting of three blocks, we have \( \sum_{\pi=(a)(b)(cd)} \prod_{B \in \pi} \text{cum}(Y_i : i \in B) \rho_\pi = \mu_Y^3(3 + 3\alpha^k) \), and finally, \( \rho_{(1)(2)(3)(4)} = \mu_Y^4 \). Due to the symmetry of the expressions, the same calculations follow for \( E[Y_0^3 Y_k^3] \), and they also hold for \( k = 0 \). We thus conclude that

\[
\tau_{1,3} = E[Y_0^4] - \mu_Y^2 - 3\mu_Y^4 + \mu_Y^2 \left( \sum_{k=1}^{\infty} E[Y_0^2 Y_k] + E[Y_0 Y_k^2] - \frac{\mu_Y^3}{2} - 2\mu_Y^2 \right)
\]

\[
= \mu_Y \left( 1 + 6\mu_Y + 3\mu_Y^2 \right) \left( 1 + 2 \sum_{k=1}^{\infty} \alpha^k \right) = \mu_Y \left( 1 + 6\mu_Y + 3\mu_Y^2 \right) \frac{1 + \alpha}{1 - \alpha}.
\]

For \( \tau_{2,3} \), we proceed analogously, denoting the summands of \( E[Y_0^2 Y_k^2] \) by \( \rho_\pi^\prime \). Obviously, \( \rho_{(1234)}^\prime = \mu_Y \alpha^k \). Further, \( \sum_{\pi=(a)(b)(cde)} \rho_\pi^\prime = \mu_Y^2(5\alpha^k) \) and \( \sum_{\pi=(a)(bc)(de)} \rho_\pi^\prime = \mu_Y^2(1 + 3\alpha^k + 6\alpha^2k) \). For the partitions consisting of three blocks, we have \( \sum_{\pi=(a)(bc)(de)} \rho_\pi^\prime = \mu_Y^2(1 + 9\alpha^k) \) and \( \sum_{\pi=(a)(b)(c)(de)} \rho_\pi^\prime = \mu_Y^2(3 + 6\alpha^k + 6\alpha^2k) \). Next, we have \( \sum_{\pi=(a)(b)(c)(de)} \rho_\pi^\prime = \mu_Y^2(4 + 6k) \), and \( \rho_{(1)(2)(3)(4)(5)}^\prime = \mu_Y^5 \). Due to the symmetry of the expressions, the same calculations hold for the expectation \( E[Y_0^2 Y_k^3] \), and they also hold for \( k = 0 \). The assertion for this entry now follows from easy calculations, analogous to the case above.
For $\tau_{3,3}$, we proceed analogously, denoting the summands of $E[Y_0^3 Y_0^3] = \sum_{\pi \in \Pi_6} \prod_{B \in \pi} \text{cum}(Y_i; i \in B)$ by $\rho''_{\pi}$. We have $\rho''_{(123456)} = \mu Y \alpha^k$. Further, $\sum_{\pi=(a)(b)(c)(d)(e)(f)} \rho''_{\pi} = \mu^5_1 (6 \alpha^k + 9 \alpha^2k)$ and $\sum_{\pi=(a)(b)(c)(d)(e)(f)} \rho''_{(a)(b)(c)(d)(e)(f)} = \mu^2_1 (1 + 9 \alpha^2k)$. For the partitions consisting of three blocks, we have $\sum_{\pi=(a)(b)(c)(d)(e)(f)} \rho''_{\pi} = \mu^3_1 (15 \alpha^k)$, $\sum_{\pi=(a)(b)(c)(d)(e)(f)} \rho''_{\pi} = \mu^3_1 (6 + 18 \alpha^k + 36 \alpha^2k)$ and $\sum_{\pi=(a)(b)(c)(d)(e)(f)} \rho''_{\pi} = \mu^3_1 (9 \alpha^k + 6 \alpha^2k)$. Next, we consider the partitions containing four blocks, i.e., $\sum_{\pi=(a)(b)(c)(d)(e)(f)} \rho''_{\pi} = \mu^4_1 (2 + 18 \alpha^k)$ and $\sum_{\pi=(a)(b)(c)(d)(e)(f)} \rho''_{\pi} = \mu^4_1 (9 + 18 \alpha^k + 18 \alpha^2k)$. For the penultimate case, we find $\sum_{\pi=(a)(b)(c)(d)(e)(f)} \rho''_{\pi} = \mu^5_1 (6 + 9 \alpha^k)$ and finally $\rho''_{(1)(2)(3)(4)(5)(6)} = \mu^6_1$. This argumentation extends to the case $k = 0$, and easy calculations conclude the proof.

References