Supplementary material for “Efficient Estimation for Marginal Generalized Partially Linear Single-index Models with Longitudinal Data”

Peirong Xu¹, Jun Zhang¹, Xingfang Huang¹ and Tao Wang³

¹Department of Mathematics, Southeast University, Nanjing, China
²Shen Zhen-Hong Kong Joint Research Center for Applied Statistical Sciences, Institute of Statistical Sciences at Shenzhen University, College of Mathematics and Computational Science, Shenzhen University, Shenzhen, China
³Department of Biostatistics, Yale School of Public Health, Yale University, New Haven, USA

We first introduce some notations. Let \( v^k_l \) be the \((k, l)\)th element of \( V^{-1} \). Define

\[
W_2(u) = \frac{1}{n} \sum_{i=1}^{n} \sum_{k=1}^{m_i} E\{\Delta_{ikk}^2 v^k_i | U_{ik} = u\} f_{ik}(u),
\]

\[
Q(u, v) = \frac{1}{n} \sum_{i=1}^{n} \sum_{k \neq l \neq k, l=1} E[\Delta_{ikk} v^k_i | U_{il} = U_{il} = v] f_{ikl}(u, v),
\]

\[
B(B; u, v) = -\frac{1}{n} \sum_{i=1}^{n} \sum_{k \neq l \neq k, l=1} E[\Delta_{ikk} v^k_i | U_{il} = u] B(U_{il}, v) f_{ikl}(u, v).
\]

Let \( \hat{\gamma}_V = (\hat{\beta}^r_V^T, \hat{\alpha}_V^T)^T \) and \( \gamma_0 = (\beta^r_0^T, \alpha_0^T)^T \). Let \( \mu_{ik}(\hat{\beta}^r_V, \hat{\alpha}_V) = g(\hat{\theta}_V(U_{ik}, \hat{\beta}^r_V, \hat{\alpha}_V) + Z_{ik}^T \tilde{\alpha}_V), \)

\[
\mu_{ik}(\beta^r_0, \alpha_0) = g(\theta_0(U_{ik}, \beta^r_0, \alpha_0) + Z_{ik}^T \alpha_0),
\]

and similar as \( \mu^{(1)}_{ik}(\beta^r_0, \alpha_0) \).

In order to establish Theorem 1, we need the following lemma first, which can be derived using similar arguments as the proof of Theorem 1 of Xu and Zhu (2012). The details are omitted here.

**Lemma 0.1.** Let \( \hat{\theta}_V(u, \beta^r_0, \alpha_0) \) be the estimate via (4). Under conditions C1-C3 and condition C7, we have

\[
\hat{\theta}_V(u, \beta^r_0, \alpha_0) - \theta_0(u, \beta^r_0, \alpha_0)
\]

\[
= W_2^{-1}(u) \frac{1}{n} \sum_{i=1}^{n} \sum_{k=1}^{m_i} \mu^{(1)}_{ik}(\beta^r_0, \alpha_0) K_{ik}(U_{ik} - u) \left\{ \sum_{l=1}^{m_i} v^k_l(Y_{il} - \mu_{il}(\beta^r_0, \alpha_0)) \right\}
\]

\[
+ W_2^{-1}(u) \frac{1}{n} \sum_{i=1}^{n} \sum_{k=1}^{m_i} \mu_{ik}(\beta^r_0, \alpha_0) Q_{1, *}(u, U_{ik}) \left\{ \sum_{l=1}^{m_i} v^k_l(Y_{il} - \mu_{il}(\beta^r_0, \alpha_0)) \right\}
\]

\[
+ W_2^{-1}(u) \frac{1}{n} \sum_{i=1}^{n} \sum_{k=1}^{m_i} v^k_l Q_{2, *}(u, U_{ik})(Y_{ik} - \mu_{ik}(\beta^r_0, \alpha_0))
\]

\[
\frac{1}{2} b_s(u) h^2 + o_p(h^2 + (\log n/nh)^{1/2} + n^{-1/2}),
\]

where \( b_s(u) = \hat{\theta}_0^{(2)}(u) - W_2^{-1}(u) \frac{1}{n} \sum_{i=1}^{n} \sum_{k \neq l \neq k, l=1} E\{\Delta_{ikk} v^k_i | U_{il} = u\} f_{ik}(u), \)

\( Q_{1, *}(u, v) = -Q(u, v) + B(Q_{1, *}, u, v), \)

and \( Q_{2, *}(u, v) = B(Q_{2, *}, u, v). \)

**Proof of Theorem 1.** The theorem can be completed following the spirit of Step 2 in the proof of Theorem 1 of Xu and Zhu (2012). We just outline some key steps here.

An application of Taylor expansion yields that

\[
\hat{\mu}_{ik}(\hat{\beta}^r_V, \hat{\alpha}_V) - \mu_{ik}(\beta^r_0, \alpha_0)
\]

\[
= \mu^{(1)}_{ik}(\beta^r_0, \alpha_0)(\hat{X}_{ik}^T, \hat{Z}_{ik}^T)(\hat{\gamma}_V - \gamma_0)
\]

\[
+ \mu^{(1)}_{ik}(\beta^r_0, \alpha_0)\{\hat{\theta}_V(U_{ik}, \beta^r_0, \alpha_0) - \theta_0(U_{ik}, \beta^r_0, \alpha_0)\} + o_p(1).
\]

¹ Corresponding Author: Peirong Xu (xupeirong@seu.edu.cn).
For simplicity, let $\hat{\theta}_V(U_{ik}) = \hat{\theta}_V(U_{ik}, \beta_0^{(r)}, \alpha_0)$ and $\theta_0(U_{ik}) = \theta_0(U_{ik}, \beta_0^{(r)}, \alpha_0)$. Denote $\hat{\mu}_i(\beta_V^{(r)}, \alpha_V) = (\hat{\mu}_i(\beta_V^{(r)}, \alpha_V), \ldots, \hat{\mu}_i m(\beta_V^{(r)}, \alpha_V))^T$ and similarly as $\mu_i(\beta_0^{(r)}, \alpha_0), \mu_i(\beta_0^{(r)}, \alpha_0), \hat{\theta}_V(U_i)$ and $\theta_0(U_i)$. Let $\mathbf{1}_m$ be an $m$-dimensional vector of ones, and denote $a \ast b$ the elementwise product of vectors $a$ and $b$. Then, putting the above equation in a matrix form and together with (5), we can derive that

$$\frac{1}{n} \sum_{i=1}^{n} A_i \sqrt{n}(\hat{\gamma} - \gamma_0) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{\partial \hat{\mu}_i(\beta_0^{(r)}, \alpha_0)}{\partial (\beta_0^{(r)}, \alpha_0)^T} V_i^{-1} [Y_i - \mu_i(\beta_0^{(r)}, \alpha_0)] - \frac{1}{\sqrt{n}} \sum_{i=1}^{n} B_i + o_p(1_K),$$

where $K = p - 1 + q$, and

$$A_i = \{\mu_i(\beta_0^{(r)}, \alpha_0) * (\hat{X_i}, \hat{Z_i})\}^T V_i^{-1} \{\mu_i(\beta_0^{(r)}, \alpha_0) * (\hat{X_i}, \hat{Z_i})\},$$

$$B_i = \{\mu_i(\beta_0^{(r)}, \alpha_0) * (\hat{X_i}, \hat{Z_i})\}^T V_i^{-1} [\mu_i(\beta_0^{(r)}, \alpha_0) * (\hat{\theta}_V(U_i) - \theta_0(U_i))].$$

Consequently, by Lemma 0.1, we have

$$\sqrt{n}(\hat{\gamma} - \gamma_0) = [A(V)]^{-1}(S_n - T_n),$$

where

$$S_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (\hat{X_i}, \hat{Z_i})^T D_{ii} V_i^{-1} \{Y_i - \mu_i(\beta_0^{(r)}, \alpha_0)\},$$

$$T_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} B_i = (T_{1n} + T_{2n})[1 + o_p(1_K)],$$

where

$$T_{1n} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \sum_{k=1}^{m} \sum_{l=1}^{m_i} \mu_i(\beta_0^{(r)}, \alpha_0) v_{il}^{(1)}(\hat{X}_{ik}, \hat{Z}_{ik})[2^{-1} h^2 \{b_*(U_{il}) + h h_{11}(U_{il}) + O_p(h^2)]],$$

$$T_{2n} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \sum_{k=1}^{m} \sum_{l=1}^{m_i} \mu_i(\beta_0^{(r)}, \alpha_0) v_{il}^{(1)}(\hat{X}_{ik}, \hat{Z}_{ik}) \times \left[ W_2^{-1}(U_{il}) \sum_{s=1}^{m} \sum_{t=1}^{m} \mu_{st}(\beta_0^{(r)}, \alpha_0) \left\{ K_h (U_{st} - U_{il}) \sum_{d=1}^{m_i} v_{s}^{(1)} (Y_{sd} - \mu_{sd}(\beta_0^{(r)}, \alpha_0)) Q_{1,s}(U_{il}, U_{st}) \right\} \right].$$

with $b_{s+1}$ being the next order term in a higher order bias expansion of $\hat{\theta}_V$. Using similar arguments as Step 2 in the proof of Theorem 1 of Xu and Zhu (2012), we can conclude that $T_{1n} = o_p(1_K)$ and $T_{2n} = o_p(1_K)$ as $n \to \infty$, $h \to 0$ such that $nh^8 \to 0$ and $nh/\log(1/h) \to \infty$, which results in $T_n = o_p(1_K)$. Consequently, based on (0.1), we have

$$\sqrt{n}(\hat{\gamma} - \gamma_0) = [A(V)]^{-1} S_n [1 + o_p(1_K)] \overset{L}{\to} N(0, [A(V)]^{-1} B(V, \Sigma) [A(V)]^{-1}),$$

which finishes the proof.

**Proof of Lemma 1.** To prove Lemma 1, the following result in Huang et al. (2007) will be needed. We reproduce it here for the sake of readability.
Lemma 0.2. For the restricted moment model

\[ Y_i = \mu(X_i, \beta) + \epsilon_i, \quad E(\epsilon_i | X_i) = 0, \]

where \( \mu(\cdot) \) is a known function, \( Y_i = (Y_{i1}, \ldots, Y_{im})^T \), \( X_i \) is the corresponding covariate matrix, and \( \epsilon_i = (\epsilon_{i1}, \ldots, \epsilon_{im})^T \). The semiparametric efficient score for \( \beta \) is

\[ S_{\text{seff}} = \sum_{i=1}^{n} \left( \frac{\partial \mu(X_i, \beta)}{\partial \beta} \right)^T \left( E(\epsilon_i \epsilon_i^T | X_i) \right)^{-1} \epsilon_i. \]

Now let \( \mathcal{P} \) be the model specified by (1). We define the following submodels:

\[ \mathcal{P}_1 = \{ \text{Model } \mathcal{P} \text{ with only } \gamma_0 \text{ unknown} \}, \]
\[ \mathcal{P}_2 = \{ \text{Model } \mathcal{P} \text{ with only } \theta_0(\cdot) \text{ unknown} \}, \]
\[ \mathcal{P}_3 = \{ \text{Model } \mathcal{P} \text{ with both } \gamma_0 \text{ and } \theta_0(\cdot) \text{ unknown} \}. \]

Denote by \( \mathcal{P}_1, \mathcal{P}_2, \) and \( \mathcal{P}_3 \) the tangent spaces corresponding to these submodels. Let \( S_\gamma \) be the score function for \( \gamma_0 \) in Model \( \mathcal{P}_1 \). Then, the semiparametric efficient score for \( \gamma_0 \) is defined as

\[ S_{\text{seff}} = S_\gamma - \Pi(S_\gamma | \mathcal{P}_3) \]
\[ = S_\gamma - \Pi(S_\gamma | \mathcal{P}_3) - \Pi\{S_\gamma | \Pi(\mathcal{P}_2 | \mathcal{P}_3)\}, \]

according to Definition 2 in Section 4.4 of Tsiatis (2006). Let \( \mathcal{P} = \{ \text{Model } \mathcal{P} \text{ with } \theta_0(\cdot) \text{ known} \}. \) Then, the semiparametric efficient score for \( \gamma_0 \) in Model \( \mathcal{P} \) is

\[ S_\gamma - \Pi(S_\gamma | \mathcal{P}_3). \]

According to Lemma 0.2, we have

\[ S_\gamma - \Pi(S_\gamma | \mathcal{P}_3) = \sum_{i=1}^{n} \left( X_i^*, Z_i \right)^T \Delta_{i0} \Sigma_i^{-1} \{ Y_i - g(\theta_0(X_i, \beta_0) + Z_0 \alpha_0) \}. \]

On the other hand, by considering parametric submodels of \( \mathcal{P}_2 \) with \( \theta_0(\cdot) \) replaced by \( \theta(\xi, \cdot) \) and applying Lemma 0.2, we can show that

\[ \Pi(\mathcal{P}_2 | \mathcal{P}_3) = \left\{ \sum_{i=1}^{n} \left[ \varphi(X_i, \beta_0) \right]^T \Delta_{i0} \Sigma_i^{-1} \{ Y_i - g(\theta_0(X_i, \beta_0) + Z_0 \alpha_0) \} : \varphi(\cdot) \in \mathcal{L}_2(\mathcal{U}) \right\}. \]

Consequently,

\[ S_{\text{seff}} = \sum_{i=1}^{n} E[(\tilde{X}_i^*, \tilde{Z}_i)^T \Delta_{i0} \Sigma_i^{-1} \{ Y_i - g(\theta_0(X_i, \beta_0) + Z_0 \alpha_0) \}], \]

where \( (\varphi_\beta^*(X_i, \beta_0), \varphi_\alpha^*(X_i, \beta_0)) \) satisfies the requirement that \( S_{\text{seff}} \) is orthogonal to any member in \( \Pi(\mathcal{P}_2 | \mathcal{P}_3) \).

That is, \( \varphi_\beta^*(X_i, \beta_0) = (\varphi_\beta^{*1}(X_i, \beta_0), \ldots, \varphi_\beta^{*p-1}(X_i, \beta_0)) \) with \( \varphi_\beta^{*p}(\cdot) \in \mathcal{L}_2(\mathcal{U}) \) needs to satisfy

\[ \sum_{i=1}^{n} E[(x_{ij} - \varphi_\beta^{*j}(X_i, \beta_0))^T \Delta_{i0} \Sigma_i^{-1} \Delta_{i0} \varphi(X_i, \beta_0)] = 0, \quad \forall \varphi(\cdot) \in \mathcal{L}_2(\mathcal{U}), \]

and \( \varphi_\alpha^*(X_i, \beta_0) = (\varphi_\alpha^{*1}(X_i, \beta_0), \ldots, \varphi_\alpha^{*q}(X_i, \beta_0)) \) with \( \varphi_\alpha^{*j}(\cdot) \in \mathcal{L}_2(\mathcal{U}) \) satisfies

\[ \sum_{i=1}^{n} E[(z_{ij} - \varphi_\alpha^{*j}(X_i, \beta_0))^T \Delta_{i0} \Sigma_i^{-1} \Delta_{i0} \varphi(X_i, \beta_0)] = 0, \quad \forall \varphi(\cdot) \in \mathcal{L}_2(\mathcal{U}). \]
Proof of Theorem 2. We first show that $\varphi_\beta (u) = \varphi_\beta ^* (u)$. Let $\varphi_\beta (u) = (\varphi_\beta (u), \ldots , \varphi_{\beta_{p-1}} (u))^T$. It is sufficient to show that $\varphi_\beta (u)$ satisfies the Fredholm integral equation of the second kind (6).

For any given $\beta (r)$ and $\alpha$, using (4) with $V_i = \Sigma_i$, we have

$$0 = 1 \sum_{i=1}^{n} \sum_{k=1}^{m_i} \sigma_i^{kk} K_h (U_{ik} - u) \mu^{(1)}_i (\beta (r), \alpha) \{ Y_{ik} - g (Z_{ik}^T \alpha + \tilde{\theta}_\Sigma (u, \beta (r), \alpha) + \tilde{b}_\Sigma (u, \beta (r), \alpha) (U_{ik} - u)) \} + \frac{1}{n} \sum_{i=1}^{n} \sum_{k \neq l, k, l = 1}^{m_i} \sigma_i^{kl} K_h (U_{ik} - u) \mu^{(1)}_i (\beta (r), \alpha) \times \{ Y_{il} - g (Z_{il}^T \alpha + \tilde{\theta}_\Sigma (U_{il}, \beta (r), \alpha)) \},$$

where $\mu^{(1)}_i (\beta (r), \alpha)$ is the first derivative of the function $g(\cdot)$ evaluated at $Z_{ik}^T \alpha + \tilde{\theta}_\Sigma (u, \beta (r), \alpha) + \tilde{b}_\Sigma (u, \beta (r), \alpha) (U_{ik} - u)$. Let $K_h^{(1)} (u) = h^{-1} K^{(1)} (u/h)$, where $K^{(1)} (\cdot)$ is the first derivative of the function $K(\cdot)$. Then, taking derivatives with respect to $\beta (r)$, we have

$$0 = T_1 + T_2 + T_3 - T_4,$$

where

$$T_1 = \frac{1}{n} \sum_{i=1}^{n} \sum_{k=1}^{m_i} \sigma_i^{kk} h^{-1} K_h^{(1)} (U_{ik} - u) J_{\beta (r)}^T X_i \mu^{(2)}_i (\beta (r), \alpha) \{ Y_{ik} - g (Z_{ik}^T \alpha + \tilde{\theta}_\Sigma (U_{il}, \beta (r), \alpha)) \};$$

$$T_2 = \left\{ \frac{1}{n} \sum_{i=1}^{n} \sum_{k=1}^{m_i} K_h (U_{ik} - u) \mu^{(2)}_i (\beta (r), \alpha) \left[ -\varphi_\beta (u, \beta (r), \alpha) + \frac{\partial \tilde{\theta}_\Sigma (u, \beta (r), \alpha)}{\partial \beta (r)} (U_{ik} - u) \right] \right\} + \frac{1}{n} \sum_{i=1}^{n} \sum_{k=1}^{m_i} K_h (U_{ik} - u) \mu^{(2)}_i (\beta (r), \alpha) \tilde{b}_\Sigma (u, \beta (r), \alpha) J_{\beta (r)}^T X_i \times \left\{ Y_{ik} - g (Z_{ik}^T \alpha + \tilde{\theta}_\Sigma (U_{il}, \beta (r), \alpha)) \right\} + \sum_{k \neq l, k, l = 1}^{m_i} \sigma_i^{kl} \left\{ Y_{il} - g (Z_{il}^T \alpha + \tilde{\theta}_\Sigma (U_{il}, \beta (r), \alpha)) \right\};$$

$$T_3 = \frac{1}{n} \sum_{i=1}^{n} \sum_{k=1}^{m_i} \sigma_i^{kk} K_h (U_{ik} - u) \mu^{(1)}_i (\beta (r), \alpha) \{ Y_{ik} - g (Z_{ik}^T \alpha + \tilde{\theta}_\Sigma (U_{il}, \beta (r), \alpha)) \} + \tilde{\varphi}_\beta (u, \beta (r), \alpha) + \frac{\partial \tilde{\theta}_\Sigma (u, \beta (r), \alpha)}{\partial \beta (r)} (U_{ik} - u) \times \left\{ Y_{il} - g (Z_{il}^T \alpha + \tilde{\theta}_\Sigma (U_{il}, \beta (r), \alpha)) \right\};$$

$$T_4 = \frac{1}{n} \sum_{i=1}^{n} \sum_{k \neq l, k, l = 1}^{m_i} \sigma_i^{kl} K_h (U_{ik} - u) \mu^{(1)}_i (\beta (r), \alpha) g^{(1)} (Z_{ik}^T \alpha + \tilde{\theta}_\Sigma (U_{il}, \beta (r), \alpha)) \times \left\{ Y_{il} - g (Z_{il}^T \alpha + \tilde{\theta}_\Sigma (U_{il}, \beta (r), \alpha)) \right\}.$$

We first consider $T_1$. Denote by $T_{11}$ the first term of $T_1$. Let $a_{ik} = \sigma_i^{kk} h^{-1} K_h^{(1)} (U_{ik} - u) J_{\beta (r)}^T X_i$, $e_{ik} = Y_{ik} - g (\theta_0 (X_{ik}^T \beta_0) + Z_{ik}^T \alpha_0)$, and $\tilde{\mu}_{ik} = g (Z_{ik}^T \alpha + \tilde{\theta}_\Sigma (u, \beta (r), \alpha) + \tilde{b}_\Sigma (u, \beta (r), \alpha) (U_{ik} - u))$. Then, we
decompose $T_{11}$ as follows:

$$
T_{11} = \frac{1}{n} \sum_{i=1}^{n} \sum_{k=1}^{m} a_{ik} g^{(1)}(\theta_0(X^T_{ik}\beta_0) + z_{ik}^{T}\alpha_0)\epsilon_{ik} \\
+ \frac{1}{n} \sum_{i=1}^{n} \sum_{k=1}^{m} a_{ik} \epsilon_{ik} \left\{ \mu^{(1)}_{ik}(\beta^{(r)}, \alpha) - g^{(1)}(\theta_0(X^T_{ik}\beta_0) + z_{ik}^{T}\alpha_0) \right\} \\
+ \frac{1}{n} \sum_{i=1}^{n} \sum_{k=1}^{m} a_{ik} g^{(1)}(\theta_0(X^T_{ik}\beta_0) + z_{ik}^{T}\alpha_0) \left\{ g(\theta_0(X^T_{ik}\beta_0) + z_{ik}^{T}\alpha_0) - \tilde{\mu}_{ik} \right\} \\
+ \frac{1}{n} \sum_{i=1}^{n} \sum_{k=1}^{m} a_{ik} \left\{ \mu^{(1)}_{ik}(\beta^{(r)}, \alpha) - g^{(1)}(\theta_0(X^T_{ik}\beta_0) + z_{ik}^{T}\alpha_0) \right\} \\
\times \left\{ g(\theta_0(X^T_{ik}\beta_0) + z_{ik}^{T}\alpha_0) - \tilde{\mu}_{ik} \right\} \\
\triangleq T_{111} + T_{112} + T_{113} + T_{114}.
$$

According to condition C3 and using the similar arguments to the proof of Theorem 1 in Xu and Zhu (2012), we have $T_{11l} = o_p(1)$ for $l = 1, 2, 3, 4$. Similarly, we can show that the second term of $T_1$ is also of order $o_p(1)$, which implies that $T_1 = o_p(1)$. Moreover, under conditions C1-C5, we know that $T_2 = o_p(1)$, and

$$
\frac{1}{n} \sum_{i=1}^{n} \sum_{k=1}^{m} \sigma_{i}^{kk} K_h(U_{ik} - u) \{ \mu^{(1)}_{ik}(\beta^{(r)}, \alpha) \}^2 2 \Delta_{\Sigma}(u, \beta^{(r)}, \alpha) \left( \frac{\partial}{\partial \beta^{(r)}} \right) (U_{ik} - u) = o_p(1).
$$

In addition, it is straightforward to show that

$$
\frac{1}{n} \sum_{i=1}^{n} \sum_{k=1}^{m} \sigma_{i}^{kk} K_h(U_{ik} - u) \{ \mu^{(1)}_{ik}(\beta^{(r)}, \alpha) \}^2 \Delta_{\beta^{(r)}}^2 = \frac{1}{n} \sum_{i=1}^{n} \sum_{k=1}^{m} E \left\{ \sigma_{i}^{kk} \Delta_{\beta^{(r)}}^2 | U_{ik} = u \right\} f_{ik}(u) \{ 1 + o_p(1) \},
$$

$$
= \frac{1}{n} \sum_{i=1}^{n} \sum_{k=1}^{m} \sigma_{i}^{kk} K_h(U_{ik} - u) \{ \mu^{(1)}_{ik}(\beta^{(r)}, \alpha) \}^2 J_{\beta^{(r)}}^{T} X_{ik}
$$

$$
= \frac{1}{n} \sum_{i=1}^{n} \sum_{k=1}^{m} E \left\{ \sigma_{i}^{kk} \Delta_{\beta^{(r)}}^2 J_{\beta^{(r)}}^{T} X_{ik} | U_{ik} = u \right\} f_{ik}(u) \{ 1 + o_p(1) \},
$$

$$
= \frac{1}{n} \sum_{i=1}^{n} \sum_{k=1}^{m} \sigma_{i}^{kk} K_h(U_{ik} - u) \{ \mu^{(1)}_{ik}(\beta^{(r)}, \alpha) \} g^{(1)}_{id} \Delta_{\beta^{(r)}}^{(1)} (U_{il}, \beta^{(r)}, \alpha) J_{\beta^{(r)}}^{T} X_{il}
$$

where $g^{(1)}_{id} = g^{(1)}(Z_{il}^{T} \alpha + \tilde{\beta}_{\Sigma}(U_{il}, \beta^{(r)}, \alpha))$ for short. On the other hand, we know that

$$
\frac{1}{n} \sum_{i=1}^{n} \sum_{k=1}^{m} \sigma_{i}^{kk} K_h(U_{ik} - u) \{ \mu^{(1)}_{ik}(\beta^{(r)}, \alpha) \} g^{(1)}_{il} \tilde{\beta}_{\beta}(U_{il}, \beta^{(r)}, \alpha)
$$

$$
= \frac{1}{n} \sum_{i=1}^{n} \sum_{k=1}^{m} \int E \left\{ \sigma_{i}^{kk} \Delta_{\beta^{(r)}}^2 | U_{ik} = u \right\} \tilde{\beta}_{\beta}(U_{il}, \beta^{(r)}, \alpha) f_{ik}(U_{il}, u) dU_{il} \{ 1 + o_p(1) \}.
$$

5
Therefore, combing the above results, we conclude that

\[
\alpha_p(1) = \frac{1}{n} \sum_{i=1}^{n} \sum_{k=1}^{m_i} E \left\{ \sigma^2_{ik} \Delta_{ik}^2 u_{ik} = u \right\} f_{ik}(u) \hat{\varphi}_\beta(U_{it}, \beta^{(r)}, \alpha)
\]

\[
- \frac{1}{n} \sum_{i=1}^{n} \sum_{k=1}^{m_i} E \left\{ \sigma^2_{ik} \Delta_{ik}^2 J_{(r)}^T \cdot X_{ik} | U_{ik} = u \right\} f_{ik}(u) \hat{\theta}_0(u) \hat{\theta}_0'(u) (u) \alpha
\]

\[
- \frac{1}{n} \sum_{i=1}^{n} \sum_{k\neq l, k,l=1}^{m_i} E \left\{ \sigma_{i}^{k} \Delta_{ikk} \Delta_{il} \hat{\theta}_0(U_{il}) J_{(r)}^T \cdot X_{ik} | U_{ik} = u \right\} f_{ik}(u)
\]

\[
+ \frac{1}{n} \sum_{i=1}^{n} \sum_{k\neq l, k,l=1}^{m_i} \int E \left\{ \sigma_{i}^{k} \Delta_{ikk} \Delta_{il} | U_{ik} = u \right\} \varphi_\beta(U_{it}, \beta^{(r)}, \alpha) f_{ilik}(U_{it}, u) dU_{it},
\]

uniformly on \( u \). Since \( \varphi_\beta(U_{it}, \beta^{(r)}, \alpha) \to \varphi_\beta(u) \) as \( n \to \infty \) uniformly on \( u \), \( \varphi_\beta(u) \) satisfies

\[
\sum_{i=1}^{n} \sum_{k=1}^{m_i} E \left\{ \sigma^2_{ik} \Delta_{ik}^2 u_{ik} = u \right\} f_{ik}(u) \varphi_\beta(u)
\]

\[
= \sum_{i=1}^{n} \sum_{k=1}^{m_i} E \left\{ \sigma^2_{ik} \Delta_{ik}^2 J_{(r)}^T \cdot X_{ik} | U_{ik} = u \right\} f_{ik}(u) \hat{\theta}_0(u) \hat{\theta}_0'(u) (u)
\]

\[
- \sum_{i=1}^{n} \sum_{k\neq l, k,l=1}^{m_i} E \left\{ \sigma_{i}^{k} \Delta_{ikk} \Delta_{il} \hat{\theta}_0(U_{il}) J_{(r)}^T \cdot X_{ik} | U_{ik} = u \right\} f_{ik}(u)
\]

\[
- \sum_{i=1}^{n} \sum_{k\neq l, k,l=1}^{m_i} \int E \left\{ \sigma_{i}^{k} \Delta_{ikk} \Delta_{il} | U_{ik} = u \right\} \varphi_\beta(U_{it}, \beta^{(r)}, \alpha) f_{ilik}(U_{it}, u) dU_{it}
\]

which demonstrates that \( \varphi_\beta(u) \) satisfies (6) for \( j = 1, \ldots, p - 1 \). Similar arguments can be used to prove \( \varphi_\alpha(u) \) satisfies (7), where \( \varphi_\alpha(u) = (\varphi_{\alpha_1}(u), \ldots, \varphi_{\alpha_q}(u))^T \).

**Proof of Corollary 2.** When \( V_i = \Sigma_i \) for \( i = 1, \ldots, n \), by Theorem 2, we know that \( A(\Sigma) = I_{\text{self}} \). Therefore, by Theorem 1, we have

\[
\sqrt{n} \left( \frac{\hat{\beta}_{(r)} - \beta_{(r)}}{\hat{\alpha}_{0} - \alpha_{0}} \right) \to N(0, I_{\text{self}}^{-1}),
\]

which finishes the proof.

**References**

