Appendix

The Discussion about the unique solution when $k_j = 3$.

In the appendix, we show that there is a one to one relationship in most cases between $\beta_{1j}$, $\beta_{2j}$ and $p_j$ when $k_j = 3$ and none of $p_{1j}$, $p_{2j}$ and $p_{3j}$ is 0. Without loss of generosity, the category values are specified as $-1, 0, 1$. With this expression, the relationships between $\beta_1$, $\beta_2$ and $p_j$ could be shown completely without $p_{2j}$, as in Figure 1. In addition, we could also see that the $\sum_{i_j=-1}^{1} i_j^3 p_{ij}$ and $\sum_{i_j=-1}^{1} i_j^2 p_{ij} = \sum_{i_j=-1}^{1} i_j p_{ij}$ in this expression.

According to Figure 1, we could see that the $\beta_{1j}$ is reflected by the lines $p_{1j} = p_{2j}$ and the $\beta_{2j}$ is symmetric by the lines $p_{1j} = p_{2j}$. The same result could be shown by plugging $(p_{1j}, p_{2j})$ and $(p_{2j}, p_{1j})$ into the Equation (6) and Equation (7).

To examine the uniqueness, we first show that when any of $p_{1j}$, $p_{2j}$ and $p_{3j}$ is 0, the $p_j$ is not unique with the same $\beta_{1j}$ and $\beta_{2j}$ combination. When any of them is 0, the $p_j$ could be considered as a Bernoulli distribution, which means that the $p_1 = \{0, a, b\}$, $p_2 = \{a, 0, b\}$ and $p_3 = \{a, b, 0\}$ could be considered as the same Bernoulli distribution. Therefore they have the same $\beta_{1j}$ and $\beta_{2j}$ combination.

For the conditions that none of $p_{1j}$, $p_{2j}$ and $p_{3j}$ is 0, the uniqueness is examined in the neighborhood and in the whole domain. The inverse function theorem is applied to examine the uniqueness of $p_j$ in the neighborhood. According to the theorem, the $p_j$ is unique in the neighborhood if the Jacobian determinant of $\beta_{1j}$ and $\beta_{2j}$ is non-zero. We could obtain the Jacobian determinant by plugging the Equation (13) and (14) in the manuscript, and show that the Jacobian determinant is 0 when

$$ p_{1j}^2 + p_{3j}^2 - p_{1j} - p_{3j} + 10p_{1j}p_{3j} = 0 \quad (1) $$
The Equation(1) is a quadratic function of $p_{1j}$ or $p_{3j}$ when the other one is fixed. Without lose of generosity, we solve Equation(1) by assuming $p_{1j}$ is a constant $c$. According to the root finding formula, we get

$$p_{3j} = \frac{(1 - 10c) \pm \sqrt{(1 - 10c)^2 + 4c(1 - c)}}{2} \quad (2)$$

Since $0 < c < 1$ and $0 < p_{3j} < 1$ in our condition, Equation(2) has the only solution

$$p_{3j} = \frac{(1 - 10c) + \sqrt{(1 - 10c)^2 + 4c(1 - c)}}{2} \quad (3)$$

Therefore we have only one root when $p_{1j}$ or $p_{3j}$ is fixed. In other word, there is only one possible $p_{1j}$ satisfies Equation (1) for each $p_{3j}$. In the neighborhood of the point, there are infinite solutions for the same $\beta_{1j}$ and $\beta_{2j}$. However, the neighborhood is small. The Jacobian determinant is greater than $10^{-6}$ if the $p_{1j}$ and $p_{3j}$ are added and subtracted by $10^{-6}$, except the one is close to the saddle point, \{1/6, 2/3, 1/6\} when evaluated numerically. We evaluate it by finding the $p_j$ with Jacobian determinant equals 0 with $p_{1j}$ varies from 0 to 1 in increments of 0.001.

In the whole domain analysis, we can see that the $\beta_{1j}$ decreases when $p_{1j} \to 0+, p_{3j} \to 0$ and $p_{1j} \to 0, p_{3j} \to 1$ and it increases when $p_{1j} \to 0, p_{3j} \to 0+$ and $p_{1j} \to 1, p_{3j} \to 0$ in Figure 1. In contrast, the $\beta_{2j}$ increases in all these areas. Since $\beta_{1j}$ is reflected by and $\beta_{2j}$ is symmetric by the line $p_{1j} = p_{2j}$, we could only focus on $p_{1j} \to 0+, p_{3j} \to 0$ and $p_{1j} \to 0, p_{3j} \to 1$ to evaluate the uniqueness.

We analyze the two areas via examining the $\beta_{1j}$ and $\beta_{2j}$ analytically and numerically. First, the point {1/6, 2/3, 1/6} is the only saddle point of $\beta_{1j}$ since its partial first derivatives are both 0 but it is not a saddle point in $\beta_{2j}$. We could see that in the contour plots in Figure 1. According to the plots, we evaluate the ratio
of first derivatives of $\beta_{1j}$ to $\beta_{2j}$ along the direction $<-1/6,-2/3,5/6>$ and $<-1/6,1/3,-1/6>$. The two ratios are not multiples of each other. It implies that we could find unique $p_j$ with specific $\beta_{1j}$ and $\beta_{2j}$ combination in the two areas. To confirm it, we numerically compute the $\beta_{1j}$ and $\beta_{2j}$ with different $p_j$ in increment of 0.001 and no multiple solutions with the same $\beta_{1j}$ and $\beta_{2j}$ are found in both areas.

According to the examinations of $\beta_{1j}$ and $\beta_{2j}$, we conclude that there is unique solution for three categories $p_j$ with most $\beta_{1j}$ and $\beta_{2j}$ combinations.
Figure Captions

Figure A1. The skewness and kurtosis function values for three category probability distributions
$\beta_1$ of different $p_1$ and $p_3$

$\beta_2$ of different $p_1$ and $p_3$