Supplementary material:
The supplement contains the detailed description of the Schmidt Decomposition (SD) as well as the method of executing the SD by Singular Value Decomposition (SVD). The purpose of this part is to help to understand the key steps of quantum image retrieval in the main text.

the Schmidt Decomposition (SD)

From the beginning, we describe the SD by the theorem quoted from the well-known textbook Quantum Computation and Quantum Information.[S1] The following words with italic style are excerpted from the section 2.5 The Schmidt decomposition and purifications.

Theorem 2.7: (Schmidt decomposition) Suppose $|\varphi\rangle$ is a pure state of a composite system, $AB$. Then there exist orthonormal states $|i_A\rangle$ for system $A$, and orthonormal states $|i_B\rangle$ of system $B$ such that

$$|\varphi\rangle = \sum_i \lambda_i |i_A\rangle |i_B\rangle$$  \hspace{1cm} (2.202)

Where $\lambda_i$ are non-negative real numbers satisfying $\sum_i \lambda_i^2 = 1$ known as Schmidt coefficients.

The base $|i_A\rangle$ and $|i_B\rangle$ are called the Schmidt bases for $A$ and $B$, respectively, and the number of non-zero value $\lambda_i$ is called Schmidt number for the state $|\varphi\rangle$. The Schmidt number is an important property of a composite quantum system, which in some sense quantifies the amount of entanglement between systems $A$ and $B$.

For example, the SD of Bell state $\frac{1}{\sqrt{2}} (|00\rangle + |11\rangle)$ is shown as equation (S1).

$$\frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}} |00\rangle + \frac{1}{\sqrt{2}} |11\rangle \hspace{1cm} (S1)$$

where the coefficient of Schmidt bases $\lambda_1 = \frac{1}{\sqrt{2}}$ and $\lambda_2 = \frac{1}{\sqrt{2}}$. The Schmidt number is 2, and the Schmidt bases are $\{|0\rangle, |1\rangle\}$.

The SD of another quantum state is shown as equation (S2).

$$\frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle) = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle) = |+\rangle |+\rangle \hspace{1cm} (S2)$$

Where $\lambda = 1$. The Schmidt number is 1, and the Schmidt bases are $\{|+\rangle, |-\rangle\}$. (The coefficient of $|-\rangle$ is 0.)

How to do the SD

The standard method to do the SD is as follows: first construct density matrix $\rho = |\varphi\rangle \langle \varphi |$, then compute reduced density operator $\rho^A$ and $\rho^B$ for subsystem $A$ and $B$, and then we calculate eigenvalues and eigenvectors of $\rho^A$ and $\rho^B$ respectively. At last, we can get the SD form of $|\varphi\rangle$.

This work is tedious. There is a simpler way to calculate the Schmidt Decomposition (SD) by Singular Value Decomposition (SVD). This method
is implied in the process of proof of Theorem 2.7 in the reference [S1]. In essence, this method has no difference with doing the SD by calculating the eigenvalues of the reduced density matrix. The equivalence between these two methods can be proved according to the properties of SVD.

In Matlab, using SVD to do the SD is much easier, for there is a function 
\[ U D V = \text{SVD}(A) \] can get the decomposition result of matrix \( A \) directly. For an example, there is a 3-qubit state \( |\psi\rangle = a_0|000\rangle + a_1|001\rangle + a_2|010\rangle + a_3|011\rangle + a_4|100\rangle + a_5|101\rangle + a_6|110\rangle + a_7|111\rangle \), we want to do the SD on it. First, we can rewrite it as a vector \((a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7)^T\). Then we reshape the vector to a \( 2 \times 4 \) matrix as \((a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7)^T\) according to the rule of bipartite splitting (the first qubit for subsystem A and the \( n - 1 \) qubits for subsystem B). We do the SVD on it. In Matlab, the procedure is shown as equation (S3).

\[
\begin{bmatrix}
a_0 & a_1 & a_2 & a_3 \\
\alpha_4 & \alpha_5 & \alpha_6 & \alpha_7 \\
\end{bmatrix} = \begin{bmatrix} u_0 & u_1 \\
u_2 & u_3 \\
\end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\
0 & \lambda_2 \\
\end{bmatrix} \begin{bmatrix} v_0 & v_1 & v_2 & v_3 \\
v_4 & v_5 & v_6 & v_7 \\
\end{bmatrix} = UAV \quad (S3)
\]

In standard form of SVD, \( U \) is an \( 2 \times 2 \), \( V \) is an \( 4 \times 4 \) unitary matrix, and \( \Lambda \) is a \( 2 \times 4 \) diagonal matrix. Equation (S3) is a simplified form, which omits the 0 elements in the matrix.

According to equation (S3), we can write the SD form of \( |\psi\rangle \) as equation (S4).

\[
|\psi\rangle = \lambda_1 \begin{bmatrix} u_0 \\
u_2 \\
\end{bmatrix} \begin{bmatrix} v_0 \\
v_1 \\
v_2 \\
v_3 \\
\end{bmatrix} + \lambda_2 \begin{bmatrix} u_1 \\
u_3 \\
\end{bmatrix} \begin{bmatrix} v_4 \\
v_5 \\
v_6 \\
v_7 \\
\end{bmatrix} = \lambda_1 |i_{A1}\rangle|i_{B1}\rangle + \lambda_2 |i_{A2}\rangle|i_{B2}\rangle \quad (S4)
\]

We then reshape \( |i_{B1}\rangle \) to matrix \( \begin{bmatrix} v_0 & v_1 \\
v_2 & v_3 \\
\end{bmatrix} \), repeat the above SVD process on it. So do on \( |i_{B2}\rangle \). At last we can get the final SD form of \( n \)-qubit state.

We can use another concrete example \( |\psi\rangle = \frac{1}{2} (|00\rangle + |01\rangle + |10\rangle + |11\rangle) \) to show the advantage of this method.

\[
\begin{bmatrix}
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\
\end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\
\end{bmatrix} \begin{bmatrix} 10 \\
00 \\
\end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\
\end{bmatrix} \\
\therefore |\psi\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} \\
\end{bmatrix} + 0 \begin{bmatrix} \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} \\
\end{bmatrix} = |+\rangle +
\]

References