Supplemental material

I. SYMMETRY CLASSIFICATION OF PAIRING CHANNELS

Electrons in 3D Dirac semi-metal are described by a four component bi-spinor creation operator, $\psi_{\alpha}^\dagger = \psi_{L\uparrow}^\dagger, \psi_{L\downarrow}^\dagger, \psi_{R\uparrow}^\dagger, \psi_{R\downarrow}^\dagger$, whose index $\gamma$ takes four values. Here we classify the possible local superconducting order parameters, written generally as

$$\hat{M} = \int_r \psi_{\alpha}^\dagger (r) M_{\alpha\beta} \psi_{\beta}^\dagger (r),$$

with constant antisymmetric matrix $M$ according to representations of the 3D rotation group. The representations of the rotation group therefore characterize various possible superconducting phases. Generator of rotations consists of the orbital momentum operator $L$ and the spin operator $S_i$

$$S^i = \int_r \psi_{\gamma}^\dagger (r) \Sigma^i_{\gamma\delta} \psi_{\delta} (r),$$

Due to the rotation symmetry they transform covariantly under the action of $J = L + S$. The global quantity transforms as

$$[\hat{M}, J^i] = \int_{r,r'} [\psi_{\alpha}^\dagger (r) M_{\alpha\beta} \psi_{\beta}^\dagger (r'), \psi_{\gamma}^\dagger (r') \Sigma^i_{\gamma\delta} \psi_{\delta} (r')]$$

$$= -2 \int_r \psi_{\gamma}^\dagger (r) \Sigma^i_{\gamma\delta} M_{\delta\kappa} \psi_{\kappa}^\dagger (r).$$

Out of 16 possible matrices $M$ six are antisymmetric. They transform into each other forming the following irreducible representations.

(i) a triplet of matrices $\{T_x, T_y, T_z\} = \{\beta\alpha_z, -\gamma_x\gamma_y\gamma_z, \beta\alpha_x\}$ transforms as a vector

$$[\hat{M}^T_k, J^l] = i\varepsilon_{klm} \hat{M}_{lm}^T$$

(ii) three scalar multiplets: $S_1 = i\alpha_y; S_2 = i\Sigma_y; S_3 = -i\beta\alpha_y\gamma_5$.

Which one of the condensates is realized at zero temperature is determined by the Hamiltonian.

II. MICROSCOPIC EQUATIONS FOR LOCAL PAIRING

A. Gor’kov equations

To treat the pairing the general gaussian approximation can be employed. Using the standard formalism, the Matsubara Green’s functions ($\tau$ is the Matsubara time),

$$G_{\alpha\beta} (r, \tau; r', \tau') = -\left\langle T_{\tau} \psi_{\alpha} (r, \tau) \psi_{\beta}^\dagger (r', \tau') \right\rangle;$$

$$F_{\alpha\beta} (r, \tau; r', \tau') = \left\langle T_{\tau} \psi_{\alpha}^\dagger (r, \tau) \psi_{\beta} (r', \tau') \right\rangle,$$

obey the Gor’kov equations:

$$-\frac{\partial G_{\gamma\kappa} (r, \tau; r', \tau')}{\partial \tau} - \int_{r''} \left\langle \psi_{\gamma'} \left[ \hat{K}_{\gamma\kappa} \right] \psi_{\kappa''}\right\rangle G_{\beta\kappa} (r'', \tau; r', \tau')$$

$$-g F_{\gamma\kappa} (r, \tau; r, \tau) F_{\beta\kappa}^\dagger (r, \tau; r', \tau') = \delta_{\gamma\kappa} \delta (r - r') \delta (\tau - \tau');$$

$$\frac{\partial F_{\gamma\kappa}^\dagger (r, \tau; r', \tau')}{\partial \tau} - \int_{r''} \left\langle \psi_{\gamma'} \left[ \hat{K}_{\gamma\kappa}^\dagger \right] \psi_{\kappa''}\right\rangle F_{\beta\kappa} (r'', \tau; r', \tau')$$

$$-g F_{\gamma\kappa} (r, \tau; r, \tau) G_{\beta\kappa} (r, \tau; r', \tau') = 0.$$
In the homogeneous case the Gor’kov equations for Fourier components of the Greens functions simplify considerably,

\[
  D^{-1} \Gamma_{\beta\kappa} (\omega, p) - \Delta_{\gamma\beta} F^\dagger_{\beta\kappa} (\omega, p) = \delta_{\gamma\kappa} ;
\]

\[
  D^{-1} F^\dagger_{\beta\kappa} (\omega, p) + \Delta_{\gamma\beta} G_{\beta\kappa} (\omega, p) = 0 ,
\]

where \( \omega = \pi T (2n + 1) \) is the Matsubara frequency and \( D^{-1} \Gamma = (i \omega - \mu) \delta_{\gamma\beta} + v_F p^i \alpha^i_{\alpha\beta} \).

The matrix gap function can be chosen as (\( \Delta \) real)

\[
  \hat{\Delta}_{\beta\gamma} = g F_{\beta\gamma} (0) = \Delta M_{\gamma\beta} .
\]

These equations are conveniently presented in matrix form (superscript \( t \) denotes transposed and \( I \) - the identity matrix):

\[
  D^{-1} G - \Delta F^\dagger = I ;
\]

\[
  D^{-1} F^\dagger + \Delta^* G = 0 .
\]

Solving these equations, one obtains

\[
  G^{-1} = D^{-1} + \Delta D^t \Delta^* ;
\]

\[
  F^\dagger = - D^t \Delta^* G ,
\]

with the gap function, found from the consistency condition. Now we find solutions of this equation for each of the possible superconducting phases.

**B. Triplet solution of the gap equation**

In this phase rotational symmetry is spontaneously broken simultaneously with the electric charge \( U (1) \) (global gauge invariance) symmetry. Assuming \( z \) direction of the \( p \) - wave condensate the order parameter matrix takes a form: \( \Delta = \Delta_T M_T^T = \Delta_T \alpha_x \). In this Section we use the units of \( v_F = 1, \hbar = 1 \) and the energy scale will be set by the Debye cutoff, \( T_D = 1 \), of the electron - phonon interactions, see below. The off-diagonal matrix element of the matrix gap equation, for real \( \Delta_T > 0 \) is:

\[
  \frac{1}{g} = \sum_{\omega q} \frac{\Delta_{\alpha\beta}^2 + p_\alpha^2 - p_\beta^2 + \mu^2 + \omega^2}{(\Delta_T^2 + \omega^2)^2 + (p^2 - \mu^2)^2 + 2 (p^2 + \mu^2) \omega^2 + 2 \Delta_T^2 (p_\perp^2 - p_\parallel^2 + \mu^2)} .
\]

where \( p_\perp^2 = p_x^2 + p_y^2 \). The spectrum of elementary excitations obtained from the four poles of the Greens function, see Fig.SM1, is (in physical units)

\[
  E_{\pm}^2 = \Delta_T^2 + v_F^2 p^2 + \mu^2 \pm 2 v_F \sqrt{\Delta_T^2 p_\parallel^2 + p_\parallel^2 \mu^2} .
\]

There are two nodes at \( p_x = p_y = 0, v_F p_z = \pm \sqrt{\Delta_T^2 + \mu^2} \), when the branches \( + | E_- \| \) and \( - | E_- \| \) cross, see Fig.SM1a and a section \( p_z = 0 \) in Fig.SM1b. There is also a saddle points with energy gap, \( 2 \Delta_T \) on the circle \( p_x^2 + p_y^2 = \mu^2, p_z = 0 \) see the section in the \( p_z = 0 \) direction in Fig. SM1c. The higher energy band \( E_+ \) touches the lower band at \( p = 0 \), so that there is a Dirac point for quasiparticles, see Fig. SM1d.

Integration over \( \omega \) gives using polar coordinates for \( p \) and \( x = \cos \theta, \zeta = \sqrt{\Delta_T^2 x^2 + \mu^2} \),

\[
  \frac{1}{g} = \frac{1}{8 \pi^2} \int_{p = \max[\mu - 1,0]}^{\mu+1} \int_{x = 0}^1 p^2 \zeta \left\{ \frac{\zeta + p x^2}{\sqrt{\Delta_T^2 + p^2 + \mu^2 + 2 p \zeta} + \sqrt{\Delta_T^2 + p^2 + \mu^2 - 2 p \zeta}} \right\} ,
\]

The lower bound on the momentum integration is nonzero when chemical potential \( \mu \) exceeds \( T_D \), see Fig. SM2. The integral over \( x \) was performed analytically, while the last integral was done numerically.
FIG. 1. Spectrum of triplet excitations. a. section \( p_\perp = 0 \) in b. There is also a saddle points with energy gap, c. \( 2\Delta_T \) on the circle \( p_x^2 + p_y^2 = \mu^2, p_z = 0 \) see the section in the \( p_z = 0 \) direction in d. The higher energy band \( E_+ \) touches the lower band at \( p = 0 \), so that there is a Dirac point for quasiparticles.

\[
\frac{1}{g} = \sum_{\omega p} \frac{\Delta_S^2 + p^2 + \mu^2 + \omega^2}{(\Delta_S^2 + p^2)^2 + (\mu^2 + \omega^2 + 2\Delta_S^2)(\mu^2 + \omega^2) + 2p^2(\omega^2 - \mu^2)}.
\] (14)

FIG. 2. Chemical potential in Dirac semi-metals and the phonon mediated pairing. (a) Chemical potential relative to Dirac point is smaller that typical energy of phonons, the Debye energy \( T_D \). (b) The BCS approximation limit: the chemical potential is much larger than the Debye energy \( T_D \).

C. Singlet representations

It turns out that the second singlet gives results identical to that of the first one, while the third singlet does not have a solution in the physically interesting range of parameters. Therefore we assume the order parameter in the matrix form \( \Delta = \Delta_S M_T = i\Delta_S \sigma^y . \) The relevant matrix element of the matrix gap equation, is for real \( \Delta_S \):

\[
\frac{1}{g} = \sum_{\omega p} \frac{\Delta_S^2 + p^2 + \mu^2 + \omega^2}{(\Delta_S^2 + p^2)^2 + (\mu^2 + \omega^2 + 2\Delta_S^2)(\mu^2 + \omega^2) + 2p^2(\omega^2 - \mu^2)}.
\]
Spectrum (in physical units) now is isotropic,

\[ E_{\pm}^2 = \Delta_S^2 + (v_F |p| \pm \mu)^2. \]  

(15)

Integration over \( \omega \) gives

\[ \frac{1}{g} = \mu \sum_{\mu - T_D < \epsilon < \mu + T_D} \frac{p}{r_+ r_- (r_+ - r_-)}. \]

where \( r_{\pm} = \sqrt{\Delta_S^2 + (|p| \pm \mu)^2} \), while the \( p \) integration results in:

\[ \frac{16\pi^2}{g} = \Phi (\mu + 1, \mu) - \Phi (\max [\mu - 1, 0], \mu) \]

(16)

with

\[ \Phi (p, \mu) = r_- (p + 3\mu) + r_+ (p - 3\mu) - (\Delta_S^2 - 2\mu^2) \log [(p + r_- - \mu) (p + r_+ + \mu)] \]

\[ (17) \]

The solution is presented in Fig. 2 of the paper as lines of constant chemical potential. Having found the order parameter, one has to determine what symmetry breaking is realized by comparing energies of the solutions as explained in the text.

**III. THE BCS AND THE STRONG COUPLING LIMITS**

**A. Triplet**

In several limiting cases the integrals can be performed analytically. At zero chemical potential the results are presented in Section IV, while here we list the BCS limit of \( \mu >> T_D \) and the strong coupling case of \( g\mu^2 >> 1 \), \( \Delta_T \propto g. \)

(i) In the BCS limit one has

\[ \frac{1}{g} = \frac{a_T \mu^2}{4\pi^2} \sinh^{-1} \frac{T_D}{\Delta_T}, \]

with \( a_T = 0.69 \), leading to exponential gap dependence on \( \lambda \) when it is small:

\[ \Delta_T = T_D / \sinh (1/2a_T \lambda) \simeq 2T_D e^{-1/2a_T \lambda}. \]  

(19)

(ii) In the strong coupling one obtains with solution

\[ \Delta_T = \frac{g}{12\pi^2} \left\{ \begin{array}{ll} 6\mu^2 + 2 & \text{for } \mu < 1 \\
\mu + 1 \end{array} \right. \]  

\[ (\mu + 1)^3 \text{ for } \mu > 1. \]  

(20)

Usually the local coupling does not prefer the triplet pairing and the singlet channels of coupling are realized. We therefore turn to them.

**B. Singlet**

For singlet one has

(i) BCS, \( \mu >> T_D \)

\[ \Delta_S = T_D / \sinh (1/2\lambda) \simeq 2T_D e^{-1/2\lambda}. \]

(ii) Strong coupling

\[ \Delta_S = \frac{2\lambda (T_D + \mu)^3}{3\mu^2}. \]

(21)
C. Energies

In limiting cases, one obtains expressions in closed form.

(i) BCS, $\mu > T_D$, using equations for the triplet and for the singlet, one has the energy density:

$$F_{T,S} = -\frac{a_{T,S}\mu^2 T_D}{2\pi^2 v_F^3 \hbar^3} \left( \sqrt{T_D^2 + \Delta_T^2} - T_D \right) \simeq -\frac{a_{T,S}\mu^2 T_D^2}{\pi^2 v_F^3 \hbar^3} \exp \left( -\frac{1}{a_{T,S} \lambda} \right),$$

with $a_T = 0.69$, while $a_S = 1$ and assuming $\lambda << 1$. The ratio of the two phases gives

$$\frac{F_T}{F_S} = 0.69e^{-0.45/\lambda}. \quad (25)$$

(ii) Strong coupling limit, one obtains

$$F_T = F_S = -\frac{1}{72\pi^2 v_F^3 \hbar^3} \begin{cases} 4 (3\mu^2 + T_D^2)^2 & \text{for } \mu < T_D \\ T_D^{-2} (\mu + T_D)^6 & \text{for } \mu > T_D \end{cases}. \quad (26)$$

The difference appears at order $1/g$. To summarize, in most of the parameter range shown triplet is a bit higher than that of the singlet, but the two condensates are nearly degenerate.