Mathematical Appendix to:

“Assessing Debt Sustainability in a Stochastic Environment: 200 years of Dutch Debt and Deficit Management”, van Wijnbergen en A. France


Feedback rules and variance reduction

We collapse all possible stochastic shocks into a single shock to the primary deficit $d_t$.\(^1\)

\[ b_t = d_t + (1 + r)b_{t-1} \]
\[ d_t = \bar{d}_t + \varepsilon_t \]
\[ \Rightarrow b_t = \bar{d}_t + \varepsilon_t + (1 + r)b_{t-1} \]

$\bar{d}_t$ is the non-stochastic part of the deficit process. Define $b^\tau_t$ as the (stochastic) projection of $b_t$, constructed at time $t$, for $\tau$ periods ahead. If the shocks to $d_t$ are i.i.d. with constant variance $\sigma$, the variance of this projection equals:

\[ \text{var}(b^\tau_t) = \text{var}\left(\sum_{t'=1}^{t'=\tau} \varepsilon_{t'}(1 + r)^{t'-1}\right) \]
\[ = \sigma^2 \sum_{t'=1}^{t'=\tau} (1 + r)^{2(t'-1)} \]
\[ = \sigma^2 \frac{(1 + r)^{2\tau} - 1}{(1 + r)^2 - 1} \]

(1.2)

where without loss of generality we assume a constant real rate of interest $r$. Consider next a fiscal feedback rule as proposed by Bohn (2007):

\(^1\) For the formulas to hold, $d$ and $b$ can refer to nominal or real deficits and debt, or ratios to GDP, as long as the interest rate $r$ is defined correspondingly (nominal, real or real growth adjusted respectively).
(1.3) \[ d_t = \tilde{d}_t + \varepsilon_t - \gamma \varepsilon_{t-1} \]

The primary surplus now is subject to shocks, like in (1.1), but also contains a feedback reaction whereby a fraction \( \gamma \) of the shock in the previous period is offset this period. Using the subscript “FR” and “NFR” for cases where there is respectively there is not such a feedback rule, it follows that:

\[
b_t = \tilde{d}_t + \varepsilon_t - \gamma \varepsilon_{t-1} + (1+r)b_{t-1}
\]

\[
= \tilde{d}_t + \varepsilon_t - \gamma \varepsilon_{t-1} + (1+r)(\tilde{d}_{t-1} + \varepsilon_{t-1} - \gamma \varepsilon_{t-2}) + (1+r)^2 b_{t-2}
\]

\[
= \tilde{d}_t + \varepsilon_t + (1+r-\gamma)\varepsilon_{t-1} + (1+r)(\tilde{d}_{t-1} + \varepsilon_{t-1} - \gamma \varepsilon_{t-2}) + (1+r)^2 b_{t-2}
\]

(1.4) \[
\frac{(1+r)^{\tau}-1}{r} + (1+r-\gamma)(\sum_{i=1}^{\tau}(1+r)^{-i-1}\varepsilon_i)
\]

\[
=> \text{var} b_t^{FR} = (1+r-\gamma)^2 \text{var} b_t^{NFR}
\]

Clearly a fractional offset coefficient \( r < \gamma < 1 \) leads to variance reduction. Also, (1.4) does not depend on \( \tau \), so it holds for all finite projection horizons and for asymptotic distributions.

Slightly different assumptions on timing modify the resulting formula’s. Assuming that deficit shocks come out at the beginning of the “next” period implies no interest is paid over the shock-related additional debt the first period (think of bills being paid late without interest being charged). In that case the formula is modified slightly and becomes:

\[
\text{var}(b_t^{FR}) = \text{var}(\sum_{i=1}^{\tau}\varepsilon_i(1-\gamma)(1+r)^{i-1})
\]

(1.5) \[
= (1-\gamma)^2 \sigma^2 \frac{(1+r)^{2\tau}-1}{(1+r)^2 -1}
\]

A more drastic modification that brings the rule closer to for example the requirements of the SGP would apply the feedback coefficient not to shocks to the deficit of the previous year but
to deviations from debt from a previously agreed time path. To save on tedious notation, we assume that the non-stochastic deficit is compatible with the debt target and that time starts with the economy at the target level of debt:

\[(1.6) \quad b_0 = \bar{b} \quad \frac{\bar{d}}{\bar{d}} = -rb \]

The debt feedback rule now becomes:

\[(1.7) \quad d_t = \bar{d}_t + \varepsilon_i - \gamma(b_{t-1} - \bar{b}) \]

Debt dynamics then become:

\[
b_1 = \bar{d} + \varepsilon_1 + (1+r)b_0 \\
= b_0 + \varepsilon_1 \\
b_2 = \bar{d} + \varepsilon_2 - \gamma\varepsilon_1 + (1+r)(b_0 + \varepsilon_1) \\
= b_0 + \varepsilon_2 + (1+r-\gamma)\varepsilon_{t-1} \\
b_3 = \bar{d} + \varepsilon_3 - \gamma\varepsilon_2 - \gamma(1+r-\gamma)\varepsilon_1 + (1+r)(b_0 + \varepsilon_2 + (1+r-\gamma)\varepsilon_1) \\
= b_0 + \varepsilon_3 + (1+r-\gamma)\varepsilon_2 + (1+r-\gamma)^2\varepsilon_1 \\
\vdots \\
b_t = b_0 + \sum_{i=1}^{t}(1+r-\gamma)^{t-1}\varepsilon_i \\
= \Rightarrow \text{var } b_{t}^{FR} = \sigma^2 \frac{(1+r-\gamma)^t - 1}{(1+r-\gamma)^2 - 1} \\
\text{and} \\
\lim_{t \to \infty} \text{var } b_t^{FR} = \frac{\sigma^2}{1-(1+r-\gamma)^2} \text{ for } \gamma > r \]

With this debt feedback rule the elegance of a time independent variance reduction is lost of course. But with a feedback rule tied to reduction of excessive (i.e. above target) debt, the drift towards infinite variance that exists without debt feedback rule will be reversed if at
In the simulation section of the paper we use debt feedback rule (1.7) with the non-stochastic debt projections as target (time dependent) debt levels.