A ONLINE APPENDIX (NOT FOR PUBLICATION)

A.1 BRIEF REVIEW OF THE THEORY This section briefly reviews the theory behind monotone operators as applied to DSGE models, using the results of Greenwood and Huffman (1995) (GH, hereafter). We do not convey any new theoretical results but simply demonstrate how the monotone map can be used to prove existence and uniqueness of an equilibrium. We follow GH closely because they proved existence of equilibrium in a very general setup. Moreover, as advocated by Datta et al. (2002), Datta et al. (2005), Mirman et al. (2008), the theoretical properties of the monotone map can be extended to more complex setups. Proofs of existence using monotone operators are constructive in the sense that the numerical algorithm is a byproduct of the proof, which only adds to the appeal of policy function iteration algorithm methods.

A.1.1 ECONOMIC ENVIRONMENT The economic environment is standard. The model consists of a continuum of measure one identical agents with preferences

$$E_0 \left[ \sum_{t=0}^{\infty} \beta^t U(c_t) \right],$$

where the momentary utility function is assumed to be strictly increasing, strictly concave and twice differentiable, with $U'(0) = \infty$. The production function is given by

$$y_t = F(k_t, K_t, \eta_t),$$

where output, $y_t$, is produced with the individual agent’s capital stock, $k_t$, the aggregate capital stock, $K_t$, and is subject to a random productivity shock, $\eta_t$. The productivity shock is assumed to be drawn from a Markov distribution function, $G(\eta_{t+1}|\eta_t)$, with bounded support. The production function is assumed to satisfy the Inada conditions, $\lim_{K \to 0} F_1(K, K, \eta) = \infty$, be strictly increasing and strictly concave in its first argument, and twice differentiable in its first two arguments. Moreover, GH also impose the following somewhat nonstandard assumptions:

1. $\exists \bar{K} \ni F(\bar{K}, \bar{K}, \eta) \leq \bar{K}$
2. $\forall K \in (0, \bar{K}], F_1(K, K, \eta) + F_2(K, K, \eta) \geq 0$ and $F_{11}(K, K, \eta) + F_{21}(K, K, \eta) < 0$

Assumption 1 places an upper bound on the level of output. Assumption 2 requires that the sum of the marginal products of the individual and aggregate capital stock be positive (along the equilibrium path $k = \bar{K}$). As noted by GH, these assumptions are innocuous and hold for a wide range of economies.

The agent’s dynamic programming problem is given by

$$V(k, K, \eta) = \max_{k'} \left\{ U(F(k, K, \eta) - k') + \beta \int V(k', K', \eta')dG(\eta'|\eta) \right\},$$

where aggregate capital, $K$, has the following law of motion $K' = Q(K, \eta)$. Let the optimal policy function associated with (6) be given by $k' = q(k, K, \eta)$. By standard arguments, one can derive the corresponding Euler equation

$$U'(F(k, K, \eta) - k') = \beta \int U''(F(k', K', \eta') - k'')F_1(k', K', \eta')dG(\eta'|\eta).$$
A stationary equilibrium is a pair of functions, $k' = q(k, K, \eta)$ and $K' = Q(k, \eta)$, that satisfy optimality (i.e., solves (6)) and consistency, $q(K, K, \eta) = Q(K, \eta)$. We are now able to state the main proposition of GH.

**Proposition 1** (GH, pg. 615). There exists a nontrivial stationary equilibrium for the economy described above.

The method of proof in GH follows that of Coleman (1991) and is our primary interest because it uses Euler equation iteration and properties of monotone operators. For these reasons we repeat the proof here. Let the sequence of aggregate laws of motion, $\{H^j(K, \eta)\}_{j=0}^{\infty}$, evolve according to $H^0(K, \eta) \equiv 0$, and let $H^{j+1}(K, \eta)$ for $j \geq 0$ be defined as the solution for $x$ in the Euler equation

$$U'(F(K, K, \eta) - x) = \beta \int U'(F(x, x, \eta') - H^j(x, \eta'))F_1(x, x, \eta')dG(\eta'|\eta).$$  \hspace{1cm} (7)

(7) defines a sequential operator mapping $H^j$ into $H^{j+1}$. GH show that the left-hand side of (7) is strictly increasing in $x$, while the right-hand side is strictly decreasing in $x$. This monotonic mapping along with assumptions 1 and 2 imply the existence of a solution to (7).

The intuition behind the result is straightforward: the sequence $\{H^j(K, \eta)\}_{j=0}^{\infty}$ produces a monotonically increasing sequence for the aggregate capital stock, which is bounded above by $K$. GH prove that the pointwise limit of this sequence of functions is the aggregate policy function $\lim_{j \to \infty} H^j(K, \eta) = Q(K, \eta)$ and that the aggregate law of motion is nondegenerate (i.e., a degenerate law of motion is one that satisfies $Q(K, \eta) = F(K, K, \eta)$ for all $K$ and $\eta$).

This mapping serves as the basis for numerical algorithms discussed in this paper, among many others [Coleman (1991); Baxter (1991); Baxter et al. (1990); Davig (2004); Davig et al. (2010); Bi (2012)]. While the purpose of this paper is to provide resources to reduce the cost of implementing the computational algorithm down, the theoretical aspect of the monotone map is very appealing.

### A.2 Linear Interpolation/Extrapolation

To get an idea of how linear interpolation and extrapolation works, consider the following example with two state variables, $x_1$ and $x_2$. The nearest perimeter around the point, $(x'_1, x'_2)$, is formed by the four points $(x_{1,i}, x_{2,i})$, $(x_{1,i+1}, x_{2,i})$, $(x_{1,i}, x_{2,i+1})$, and $(x_{1,i+1}, x_{2,i+1})$, where $i$ signifies the position on the grid. We want the policy function value, $f(x'_1, x'_2)$, but we only have policy function values for the four nearest points on the grid (off the grid, we extrapolate using the nearest four points that form a square on the edge of the state space). First, holding $x_2$ fixed, interpolate/extrapolate in the $x_1$ direction to obtain

$$f(x'_1, x_{2,i}) = f(x_{1,i}, x_{2,i}) + (x'_1 - x_{1,i}) \frac{f(x_{1,i+1}, x_{2,i}) - f(x_{1,i}, x_{2,i})}{x_{1,i+1} - x_{1,i}}$$

$$= \frac{x_{1,i+1} - x'_1}{x_{1,i+1} - x_{1,i}} f(x_{1,i}, x_{2,i}) + \frac{x'_1 - x_{1,i}}{x_{1,i+1} - x_{1,i}} f(x_{1,i+1}, x_{2,i})$$  \hspace{1cm} (8)

$$f(x'_1, x_{2,i+1}) = \frac{x_{1,i+1} - x'_1}{x_{1,i+1} - x_{1,i}} f(x_{1,i}, x_{2,i+1}) + \frac{x'_1 - x_{1,i}}{x_{1,i+1} - x_{1,i}} f(x_{1,i+1}, x_{2,i+1})$$  \hspace{1cm} (9)

\[^{14}\text{In order to prove the right-hand side is strictly decreasing, the additional assumption, } 0 \leq \partial H^j(K, \eta)/\partial K \leq [F_1(K, K, \eta) + F_2(K, K, \eta)], \text{ needs to be imposed.}\]
Then interpolate/extrapolate in the \( x_2 \) direction to obtain

\[
 f(x_1', x_2') = f(x_1', x_2, i) + (x_2' - x_2, i) \frac{f(x_1', x_2, i+1) - f(x_1', x_2, i)}{x_2, i+1 - x_2, i} \\
= \frac{x_2, i+1 - x_2'}{x_2, i+1 - x_2, i} f(x_1', x_2, i) + \frac{x_2' - x_2, i}{x_2, i+1 - x_2, i} f(x_1', x_2, i+1). 
\]

(10)

Combining (8), (9) and (10) yields

\[
 f(x_1', x_2') = \omega_{2, i} f(x_1', x_2, i) + \omega_{1, i+1} f(x_1', x_2, i+1) + \omega_{1, i+1} \omega_{2, i} f(x_1', x_2, i) + \omega_{1, i+1} \omega_{2, i+1} f(x_1', x_2, i+1) \\
= \sum_{j_1=0}^{1} \sum_{j_2=0}^{1} \omega_{1, i+j_1} \omega_{2, i+j_2} f(x_1', x_2, i+j_2),
\]

which can be easily extended to any number of state variables. We assume that the points for any one dimension in the state space are uniformly spaced, which simplifies evaluation of the policy functions. If unevenly spaced nodes are desired, then Fallterp must be modified to correctly locate the nearest nodes.

### A.3 Integration

A model with both continuous and discrete stochastic variables requires two types of numerical integration. For continuous stochastic variables we apply either the Trapezoid rule or Gauss-Hermite quadrature, and for discrete random variables we use the corresponding transition matrix to weight each outcome by its likelihood.

#### A.3.1 Trapezoid Rule

Suppose there are \( m \) realizations of the stochastic component, \( \varepsilon \), in the process for some continuous variable \( z \). Since these realizations show up in agents’ expectations, we perform numerical integration to average across each of these \( m \) realizations. The trapezoid rule is one method of numerical integration. Assuming uniformly spaced realizations of \( \varepsilon \), the formula for the trapezoid rule is given by

\[
 E_i[\Phi(\cdot, z_{t+1})] \approx \frac{1}{2} \left[ \sum_{i=1}^{m} \Pr(\varepsilon_i) \Phi(\cdot, z_{t+1}(\varepsilon_i)) - \Pr(\varepsilon_1) \Phi(\cdot, z_{t+1}(\varepsilon_1)) - \Pr(\varepsilon_m) \Phi(\cdot, z_{t+1}(\varepsilon_m)) \right] \Delta \varepsilon
\]

where \( \Delta \varepsilon \) is the distance between stochastic realizations, \( \Pr(\varepsilon_i) \) is the probability of realization \( i \), and \( \Phi \) is the value of the contents of the expectation operator, given the state of the economy. To obtain the weights (the probabilities) in the trapezoid rule, truncate the distribution of the stochastic variable. For normal random variables, we recommend truncating the distribution at no less than four standard deviations, since omitting more of the distribution often leads to inaccurate results.
A.3.2 GAUSS-HERMITE QUADRATURE Another commonly employed method of numerical integration is Gauss-Hermite quadrature. Suppose a shock, \( u \), to a continuous variable, \( z \), is normally distributed with mean \( \mu \) and variance \( \sigma^2 \). Then expectations can be written as

\[
E_t[\Phi_{t+1}(\cdot, z_{t+1}(u))] = (2\pi\sigma^2)^{-1/2} \int_{-\infty}^{\infty} \Phi_{t+1}(\cdot, z_{t+1}(u)) e^{-(u-\mu)^2/(2\sigma^2)} du.
\]

Applying the change of variables, \( \varepsilon = (u-\mu)/(\sqrt{2}\sigma) \), the Gauss-Hermite quadrature rule is

\[
E_t[\Phi_{t+1}(\cdot, z_{t+1}(u))] = \pi^{-1/2} \int_{-\infty}^{\infty} \Phi_{t+1}(\cdot, z_{t+1}(\sqrt{2}\sigma \varepsilon + \mu)) e^{-\varepsilon^2} d\varepsilon
\]

\[
\approx \pi^{-1/2} \sum_{i=1}^{n} \omega_i \Phi_{t+1}(\cdot, z_{t+1}(\sqrt{2}\sigma \varepsilon_i + \mu)),
\]

where \( \varepsilon_i \) are the realizations of the standard normal shock, \( \Phi \) is the value of the contents of the expectation operator, and \( \omega_i \) are Gauss-Hermite weights given by \( \omega_i = 2^{n+1} n! \sqrt{\pi} [H_{n+1}(\varepsilon_i)]^{-2} \). \( H_{n+1} \) is the physicists’ Hermite polynomial of order \( n + 1 \).\(^{15}\)

We usually adopt the Trapezoid rule over Gauss-Hermite quadrature because it is more stable. Moreover, with dense and wide enough grids (at least 10 points and 4 standard deviations) for the continuous shocks, the optimal policy functions under these two methods of numerical integration are virtually identical, even though the Trapezoid rule relies on a truncated distribution.

A.3.3 MARKOV CHAIN INTEGRATION Suppose a discrete stochastic variable, \( z \), evolves according to an \( m \)-state first-order Markov chain.\(^{16}\) Once again, these realizations show up in agents’ expectations, and we must integrate across these \( m \) realizations conditional on the previous state. Suppose the transition matrix is given by

\[
P = \begin{bmatrix}
\Pr[s_t = 1 | s_{t-1} = 1] & \Pr[s_t = 2 | s_{t-1} = 1] & \cdots & \Pr[s_t = m | s_{t-1} = 1] \\
\Pr[s_t = 1 | s_{t-1} = 2] & \Pr[s_t = 2 | s_{t-1} = 2] & \cdots & \Pr[s_t = m | s_{t-1} = 2] \\
\vdots & \vdots & \ddots & \vdots \\
\Pr[s_t = 1 | s_{t-1} = m] & \Pr[s_t = 2 | s_{t-1} = m] & \cdots & \Pr[s_t = m | s_{t-1} = m]
\end{bmatrix} = \begin{bmatrix}
p_{11} & p_{12} & \cdots & p_{1m} \\
p_{21} & p_{22} & \cdots & p_{2m} \\
\vdots & \vdots & \ddots & \vdots \\
p_{m1} & p_{m2} & \cdots & p_{mm}
\end{bmatrix},
\]

where \( 0 \leq p_{ij} \leq 1 \) and \( \sum_{j=1}^{m} p_{ij} = 1 \) for all \( i \in \{1, 2, \ldots, m\} \). Then the conditional expectation can be written as

\[
E_t [\Phi_{t+1}(\cdot, z_{t+1}) | s_t = i] = \begin{bmatrix}
p_{i1} & p_{i2} & \cdots & p_{im}
\end{bmatrix} \begin{bmatrix}
\Phi_{t+1}(\cdot, z_{1,t+1}, s_t = i) \\
\Phi_{t+1}(\cdot, z_{2,t+1}, s_t = i) \\
\vdots \\
\Phi_{t+1}(\cdot, z_{m,t+1}, s_t = i)
\end{bmatrix}.
\]

If a model contains both continuous and discrete stochastic variables, first integrate across the continuous random variables to obtain a set of values, conditional on the realizations of the discrete stochastic variable. Then weight each of these values by their corresponding likelihood. This process yields an expected value across all stochastic components in the model.

\(^{15}\)We provide a function, ghquad.m, to compute the Gauss-Hermite weights. To calculate the coefficients of the Hermite polynomial, ghquad.m requires HermitePoly.m, which is written by David Terr and readily available on the MATLAB file exchange.

\(^{16}\)Recall that higher order Markov chains can always be described by a first-order transition matrix.