Online Appendix:
Sovereign Debt and Bank Fragility in Spain*:
A Calibrated DSGE Model

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1 Derivations

1.1 Households

The household sector is modelled as a continuum of identical, infinitely lived households. Households provide labour and engage in consumption and saving. Saving takes place by placing deposits at financial intermediaries. Each household consists of both workers and bankers. A fraction $1 - f$ are workers and earn wages that are transferred to the household. A fraction $f$ are bankers managing a financial intermediary from which the dividends are also transferred to the household. Each period a banker has a probability of $\theta$ to remain a banker in the next period. With probability $1 - \theta$ the banker is forced to exit the financial sector, and transfers the net worth of his financial intermediary to the household and becomes a worker. The income from both bankers and workers are pooled, and equally divided among household members. The household thus obtains income from supplying labor, repayment of principal and interest on previous period deposits, and profits from both financial and non-financial firms owned by the household. The funds are used to place new deposits, consumption and lump-sum taxes to the government.

The households derive utility from consumption and disutility from providing labour. Therefore, households face the following maximisation problem:

$$\max_{\{c_{t+s}, h_{t+s}, d_{t+s}\}} \mathbb{E}_t \left[ \sum_{s=0}^{\infty} \beta^s \left( \log (c_{t+s} - \psi c_t) - \frac{h_{t+s}^{1+\varphi}}{1+\varphi} \right) \right]$$

s.t. $c_t + d_t + \tau_t = w_t h_t + (1 + r_t^n) d_{t-1} + \Pi_t$,
where \( c_t \) is household consumption, \( h_t \) labour supply, and \( w_t \) the wage rate for labour. \( d_{t-1} \) are household deposits from the previous period over which net real interest rate \( r^d_{t} \) is received, \( \tau_t \) is a lump sum tax that is paid to the government and \( \Pi_t \) are profits from both financial and non-financial firms owned by the household.

This maximization problem results in the following first order conditions for the household:

\[
\begin{align*}
\lambda_t &= (c_t - vcc_{t-1})^{-1} - v\beta E_t \left[ (c_{t+1} - vcc_t)^{-1} \right], \\
\Psi h_t^\tau &= \lambda_t w_t, \\
1 &= \beta E_t \left[ \Lambda_{t,t+1}(1 + r^d_{t+1}) \right],
\end{align*}
\]

with \( \Lambda_{t,t+i} \) as the stochastic discount factor \( \lambda_{t+i}/\lambda_t \) for \( i \geq 0 \).

### 1.2 Financial Sector Derivations

Financial intermediaries hold assets in the form of claims issued by intermediate goods producers, government bonds and external bonds. External bonds are placed at the financial intermediaries in case of a recapitalization, and can not be traded in the secondary market. The assets are financed through household deposits and net worth. The balance sheet of the financial intermediary is given by

\[ p_{j,t} = n_{j,t} + d_{j,t}, \]

where \( n_{j,t} \) represents the intermediary’s net worth, \( d_{j,t} \) funds deposited by households and \( p_{j,t} \) the assets held by the intermediary. The asset side of the balance sheet is represented by:

\[ p_{j,t} = q^k_{t} s^k_{j,t} + q^b_{t} s^b_{j,t} + s^e_{j,t}, \]

with \( q^k_{t} \) and \( q^b_{t} \) denoting the price of claims on intermediate goods producers, respectively government bonds; \( s^k_{j,t} \) and \( s^b_{j,t} \) denote the amount of claims respectively bonds held, while \( s^e_{j,t} \) represents the number of external bonds.

In period \( t + 1 \), households obtain a real return of \( r^d_{t+1} \) from the financial intermediary on funds deposited in period \( t \). In turn, the intermediary receives a state-contingent net real return of \( r^k_{t+1} \) on the loans to the intermediate goods producers and \( r^b_{t+1} \) on government bonds. External bonds pay a return \( r^e_{t} \), which is determined in period \( t \) and paid out in period \( t + 1 \). Incentives for financial intermediaries to engage in banking activities follow from the possibility that \( r^k_{t+1}, r^b_{t+1} \) and \( r^e_{t} \) are greater than \( r^d_{t+1} \), meaning there is potential for growth in net worth. In addition, net worth can be altered by government support to the financial sector \( n^g_{j,t+1} \) and/or repayment of support from previous periods \( \bar{n}^g_{j,t+1} \). The law of motion for the intermediary’s net worth is thus as follows:

\[ n_{j,t+1} = (1 + r^k_{t+1}) s^k_{j,t} + (1 + r^b_{t+1}) s^b_{j,t} + (1 + r^e_{t}) s^e_{j,t} - (1 + r^d_{t+1}) d_{j,t} + n^g_{j,t+1} - \bar{n}^g_{j,t+1}. \]
Government support $n_{j,t}^g$ is linear in previous period net worth, as well as the tax $\tilde{\tau}_{t+1}$. Using $n_{j,t} = p_{j,t} - d_{j,t}$, $n_{j,t+1}^g = \tau_{t+1}^n n_{j,t}$, and $	ilde{n}_{j,t+1}^g = \tilde{\tau}_{t+1}^n n_{j,t}$, this can be rewritten as

$$n_{j,t+1} = (r_{t+1}^k - r_{t+1}^d) q_{k,t} s_{j,t} + (r_{t+1}^b - r_{t+1}^d) q_{b,t} s_{j,t} + (r_{t+1}^e - r_{t+1}^d) s_{j,t} + (1 + r_{t+1}^d) n_{j,t} + \tau_{t+1}^n n_{j,t}.$$

Each period, a fraction $\theta$ of bankers are allowed to continue operating, while a fraction $1 - \theta$ of bankers are forced to exit the financial sector. Financial intermediaries try to maximise expected dividends, which consist of total net worth if forced to exit the financial sector, and zero otherwise. On exiting the industry, a banker transfers the final net worth of the intermediary to its household. Therefore, each banker maximizes its expected net worth on exiting the industry, which can be given by

$$V_{j,t} = \max E_t \left\{ \beta \Lambda_{t,t+1} [(1 - \theta) n_{j,t+1} + \theta V_{j,t+1}] \right\},$$

where $\beta \Lambda_{t,t+1}$ is the stochastic discount factor of the household, since the household is the ultimate owner of the intermediary.

If there is a positive risk premium on bonds and/or loans, the intermediary will want to expand its holdings of assets indefinitely by borrowing funds from households at a rate lower than the rate of return on assets. However, the extent to which the intermediary can leverage its net worth is limited by an incentive problem as in Gertler and Karadi (2011): bankers have the possibility of diverting the assets of the intermediary. Depositors will force the intermediary into bankruptcy, but will only be able to recoup a fraction $1 - \lambda_a^t$ of the bank asset class $a = \{k, b, e\}$: the bankers will get away with a fraction $\lambda_a^t$, which is possibly time-varying. Since depositors have rational expectations, they will provide funds to the point where the gains from stealing are smaller or at most equal to the continuation value of the financial intermediary:

$$V_{j,t} = \max E_t \left\{ \beta \Lambda_{t,t+1} [(1 - \theta) n_{j,t+1} + \theta V_{j,t+1}] \right\},$$

where $\beta \Lambda_{t,t+1}$ is the stochastic discount factor of the household, since the household is the ultimate owner of the intermediary. We conjecture the solution to be of the following form, and
The optimization problem implies the following Lagrangian:

\[ \mathcal{L} = (1 + \mu_t) \left( \nu_t^k q_{t}^k s_{j,t}^k + \nu_t^b q_{t}^b b_{j,t} + \nu_t^\epsilon s_{j,t}^\epsilon + \eta_t n_{j,t} \right) - \mu_t \left( \lambda_t^k q_{t}^k s_{j,t}^k + \lambda_t^b q_{t}^b b_{j,t} + \lambda_t^\epsilon s_{j,t}^\epsilon \right). \]

where \( \mu_t \) is the Lagrangian multiplier on the balance sheet constraint of the intermediaries. We get the following first order conditions:

\[
\begin{align*}
  s_{j,t}^k & : \quad (1 + \mu_t) q_{t}^k \nu_t^k - \lambda_t^k \mu_t q_{t}^k = 0 \iff \nu_t^k = \frac{\lambda_t^k}{1 + \mu_t}, \\
  s_{j,t}^b & : \quad (1 + \mu_t) q_{t}^b \nu_t^b - \lambda_t^b \mu_t q_{t}^b = 0 \iff \nu_t^b = \frac{\lambda_t^b}{1 + \mu_t}, \\
  s_{j,t}^\epsilon & : \quad (1 + \mu_t) \nu_t^\epsilon - \lambda_t^\epsilon \mu_t = 0 \iff \nu_t^\epsilon = \frac{\lambda_t^\epsilon}{1 + \mu_t}, \\
  \mu_t & : \quad \left[ (\nu_t^k q_{t}^k s_{j,t}^k + \nu_t^b q_{t}^b b_{j,t} + \nu_t^\epsilon s_{j,t}^\epsilon + \eta_t n_{j,t}) - (\lambda_t^k q_{t}^k s_{j,t}^k + \lambda_t^b q_{t}^b b_{j,t} + \lambda_t^\epsilon s_{j,t}^\epsilon) \right] \mu_t = 0.
\end{align*}
\]

From the first order conditions we find that \( \nu_t^k = \left( \frac{\lambda_t^k}{\lambda_t^k + \mu_t} \right) \nu_t^k \) and \( \nu_t^\epsilon = \left( \frac{\lambda_t^\epsilon}{\lambda_t^k + \mu_t} \right) \nu_t^\epsilon \). Hence the incentive compatibility constraint of the bankers can be rewritten in the following way:

\[
\nu_t^k \left( q_{t}^k s_{j,t}^k + \left( \frac{\lambda_t^b}{\lambda_t^k} \right) q_{t}^b s_{j,t}^b + \left( \frac{\lambda_t^\epsilon}{\lambda_t^k} \right) s_{j,t}^\epsilon \right) + \eta_t n_{j,t} \geq \lambda_t^k \left( q_{t}^k s_{j,t}^k + \left( \frac{\lambda_t^b}{\lambda_t^k} \right) q_{t}^b s_{j,t}^b + \left( \frac{\lambda_t^\epsilon}{\lambda_t^k} \right) s_{j,t}^\epsilon \right),
\]

\[
\iff q_{t}^k s_{j,t}^k + \left( \frac{\lambda_t^b}{\lambda_t^k} \right) q_{t}^b s_{j,t}^b + \left( \frac{\lambda_t^\epsilon}{\lambda_t^k} \right) s_{j,t}^\epsilon \leq \phi_t n_{j,t},
\]

\[
\phi_t = \frac{\eta_t}{\lambda_t^k - \nu_t^k},
\]

where \( \phi_t \) can be seen as the leverage constraint of the financial intermediary. The intuition for the leverage constraint is straightforward: a higher shadow value of assets \( \nu_t^k \) implies a higher value from an additional unit of assets, which raises the continuation value of the financial intermediary, thereby making it less likely that the banker will steal. A higher shadow value of net worth \( \eta_t \) implies a higher expected profit from an additional unit of net worth, while a higher fraction \( \lambda_t^\epsilon \) implies that the banker can steal a larger fraction of assets, which induces the household to provide less funds to the banker, resulting in a lower leverage ratio everything else equal.

Substitution of the conjectured solution into the right hand side of the Bellman equation
After combining the conjectured solution (4) with (7), we find the following first order conditions:

\[ V_{j,t} = E_t \left[ \beta \Lambda_{t,t+1} \left\{ (1 - \theta) n_{j,t+1} + \theta V_{j,t+1} \right\} \right] \]

\[ = E_t \left[ \beta \Lambda_{t,t+1} \left\{ (1 - \theta) n_{j,t+1} + \theta \left[ \nu^k_{t+1} q^b_{t+1} s^b_{j,t+1} + \nu^b_{t+1} q^b_{t+1} s^b_{j,t+1} + \nu^c_{t+1} \theta V_{j,t+1} + \eta_{t+1} n_{j,t+1} \right] \right\} \right] \]

\[ = E_t \left[ \beta \Lambda_{t,t+1} \left\{ (1 - \theta) n_{j,t+1} + \theta \left[ \nu^k_{t+1} \left( q^k_{t+1} s^k_{j,t+1} + \left( \frac{\lambda^b_{t+1}}{\lambda^k_{t+1}} \right) q^b_{t+1} s^b_{j,t+1} + \left( \frac{\lambda^b_{t+1}}{\lambda^k_{t+1}} \right) s^c_{j,t+1} + \eta_{t+1} n_{j,t+1} \right) \right\} \right] \right] \]

\[ = E_t \left[ \beta \Lambda_{t,t+1} \left\{ (1 - \theta) n_{j,t+1} + \theta \left[ \nu^k_{t+1} \phi_{t+1} n_{j,t+1} + \eta_{t+1} n_{j,t+1} \right] \right\} \right] \]

We can rewrite this in the following way:

\[ V_{j,t} = E_t \left[ \Lambda_{t+1} n_{j,t+1} \right], \]

\[ \Lambda_{t+1} = \beta \Lambda_{t,t+1} \left\{ (1 - \theta) + \theta \left( \phi_{t+1} n_{j,t+1} + \eta_{t+1} \right) \right\}. \]

\( \Lambda_{t+1} \) can be thought of as a stochastic discount factor that incorporates the financial friction. Now substitute the expression for next period’s net worth into the expression above:

\[ V_{j,t} = E_t \left[ \Lambda_{t+1} n_{j,t+1} \right] \]

\[ = E_t \left[ \Lambda_{t+1} \left\{ \left( r^k_{t+1} - r^d_{t+1} \right) q^k_{t+1} s^k_{j,t} + \left( r^b_{t+1} - r^d_{t+1} \right) q^b_{t+1} s^b_{j,t} + \left( r^c_{t} - r^d_{t+1} \right) s^c_{j,t} + \left( 1 + r^a_{t+1} + \tau^a_{t+1} - \tau^a_{t+1} \right) n_{j,t} \right\} \right]. \]

(7)

After combining the conjectured solution (4) with (7), we find the following first order conditions:

\[ \nu^k_t = E_t \left[ \Lambda_{t+1} \left( r^k_{t+1} - r^d_{t+1} \right) \right], \]

\[ \nu^b_t = \left( \frac{\lambda^b_t}{\lambda^k_t} \right) \nu^k_t = E_t \left[ \Lambda_{t+1} \left( r^b_{t+1} - r^d_{t+1} \right) \right], \]

\[ \nu^c_t = \left( \frac{\lambda^c_t}{\lambda^k_t} \right) \nu^k_t = E_t \left[ \Lambda_{t+1} \left( r^c_{t} - r^d_{t+1} \right) \right], \]

\[ \eta_t = E_t \left[ \Lambda_{t+1} \left( 1 + r^d_{t+1} + \tau^a_{t+1} - \tau^a_{t+1} \right) \right], \]

with \( \Lambda_{t+1} = \beta \Lambda_{t,t+1} \left\{ (1 - \theta) + \theta \left( \nu^k_{t+1} \phi_{t+1} n_{j,t+1} + \eta_{t+1} \right) \right\}. \)

As described in section 1.1 each period a number of bankers the size of a fraction \( (1 - \theta) f \) of the households will become a worker and vice versa. The other bankers continue operating their financial intermediary. The aggregate net worth of these continuing intermediaries, abstracted from government aid and repayment, can therefore be given by

\[ n_{c,t} = \theta \left[ \left( r^k_t - r^d_t \right) q^k_{t-1} s^k_{t-1} + \left( r^b_t - r^d_t \right) q^b_{t-1} s^b_{t-1} + \left( r^c_{t-1} - r^d_t \right) s^c_{t-1} + \left( 1 + r^d_t \right) n_{t-1} \right] \]
Exiting bankers take the net worth of their intermediaries, \((1 - \theta)\) of aggregate net worth, to the household, of which a share is provided as starting capital to entering bankers. This share is \(\chi/(1 - \theta)\) of the assets held by the intermediaries of exiting bankers. Adding financial sector support by the government and repayment of support, the total net worth of the financial sector is

\[
n_t = \theta \left[ (r^k_t - r^d_t) q_{t-1}^k s_{t-1}^k + (r^b_t - r^d_t) q_{t-1}^b s_{t-1}^b + (r^e_t - r^d_t) s_{t-1} + (1 + r^d_t) n_{t-1} + \chi p_{t-1} + n^\sigma_t - \tilde{n}^\sigma_t \right] + \chi p_t - 1 + n^\eta_t - \tilde{n}^\eta_t.
\]

Finally, the number of external bonds is determined by the size of externally financed capital injections to banks:

\[
s^e_t = s^e_{t-1} + (1 - \kappa_e)(n^\sigma_t - \tilde{n}^\sigma_t).
\]

### 1.3 Production Process

#### 1.3.1 Intermediate Goods Producers

A continuum of intermediate goods producers, that face perfect competition, acquire capital \(k_{i,t-1}\) from capital producers at the end of period \(t - 1\) for a price \(q^k_{t-1}\) per unit of capital. This purchase of capital is financed by issuing \(s^k_{t-1}\) claims, equal to the number of units of capital purchased, to the financial intermediaries at price \(q^k_{t-1}\). No frictions arise in the financing of intermediate firms, hence monitoring costs are absent for intermediaries, and next period's profits can credibly be pledged to the intermediaries, as in [Gertler and Kiyotaki (2010)](https://www.nber.org/papers/w19079). After realization of the shocks, the producers hire labour \(h_{i,t}\) at a wage \(w_t\), and start producing intermediate goods with previous period capital \(k_{i,t-1}\) and labor \(h_{i,t}\) as input. After production, the intermediate goods producers pay a state-contingent net real return \(r^k_t\) over claims issued in period \(t\), where the production function produce output with the production function

\[
y_{i,t} = a_t (\xi_t k_{i,t-1})^\alpha h_{i,t}^{1-\alpha}.
\]

Quality of capital \(\xi_t\) (a shock we turn off in the main part of the model by setting \(\xi_t = 1\) for all periods) and total factor productivity \(a_t\) are driven by AR(1) processes

\[
\log \xi_t = \rho_\xi \log \xi_{t-1} + \varepsilon_{\xi,t} \quad \text{and} \quad \log a_t = \rho_a \log a_{t-1} + \varepsilon_{a,t},
\]

where \(\varepsilon_{\xi,t} \sim N(0, \sigma^2_\xi)\) and \(\varepsilon_{a,t} \sim N(0, \sigma^2_a)\) are random i.i.d. shocks. Note that the effective capital stock is influenced by the size of the capital quality shock and cannot be fully determined by the intermediate goods producer at the moment of purchase.

Output \(y_{i,t}\) is sold to retail firms for a price \(m_t\) per unit, with \(m_t = P^m_t / P_t\) representing the relative price of the intermediate goods to that of the final goods. Once the goods have
been sold, the firms sell whatever is left of the effective capital stock (after depreciation) back to the capital producers for a price $q_k^t$ and pay back the loans and a net return to the financial intermediaries. This leads to the profit function

$$\Pi_{i,t} = m_t a_t (\xi_t k_{i,t-1})^\alpha h_{i,t}^{-\alpha} + q_k^t (1 - \delta) \xi_t k_{i,t-1} - (1 + r_k^t) q_{k_{t-1}}^t k_{i,t-1} - w_t h_{i,t}. $$

The intermediate goods producing firms maximize expected current and future profits using the household’s stochastic discount factor $\beta^s \Lambda_{t,s}$ (since they are owned by the households), taking all prices as given:

$$\max_{(k_{t+s}, h_{t+s})_{s=0}} \mathbb{E}_t \left[ \sum_{s=0}^{\infty} \beta^s \Lambda_{t,s} \Pi_{i,t+s} \right].$$

The first order conditions belonging to this problem are given by:

$$k_{i,t} : \mathbb{E}_t \left[ \beta \Lambda_{t+1} \left( q_k^t (1 + r_k^t) \right) \right] = \mathbb{E}_t \left[ \beta \Lambda_{t+1} \left( \alpha m_t y_{i,t+1}/k_{i,t} + q_k^t (1 - \delta) \xi_t + 1 \right) \right],$$

$$h_{i,t} : w_t = (1 - \alpha) m_t y_{i,t}/h_{i,t}.$$

In equilibrium profits will be zero. By substituting the first order condition for the wage rate into the zero-profit condition $\Pi_{i,t} = 0$, we can find an expression for the ex-post return on capital:

$$r_k^t = \left( q_{k_{t-1}}^t \right)^{-1} \left( \alpha m_t y_{i,t}/k_{i,t-1} + q_k^t (1 - \delta) \xi_t \right) - 1.$$

Now we rewrite the first order condition for labor and the expression for the ex-post return on capital to find the factor demands:

$$k_{i,t-1} = \alpha m_t y_{i,t}/[q_k^t (1 + r_k^t) - q_k^t (1 - \delta) \xi_t],$$

$$h_{i,t} = (1 - \alpha) m_t y_{i,t}/w_t.$$

By substituting the factor demands into the production technology function, we get for the relative intermediate output price $m_t$:

$$m_t = a_t^{-\alpha} (1 - \alpha)^{-1} \xi_t^{-\alpha} \left( w_t^{1 - \alpha} g_{t-1}^k (1 + r_k^t) \xi_t^{-1} - q_k^t (1 - \delta) \right)^{\alpha}. $$

### 1.3.2 Retail firms

Retail firms purchase goods ($y_{i,t}$) from the intermediate goods producing firms for a nominal price $P_t^m$, and convert these into retail goods ($y_{f,t}$). These goods are sold for a nominal price $P_{f,t}$ to the final goods producer. It takes one intermediate goods unit to produce one retail good ($y_{i,t} = y_{f,t}$). All the retail firms produce a differentiated retail good by assumption, therefore operate in a monopolistically competitive market, and charge a markup over the input price earning them profits ($P_{f,t} - P_t^m y_{f,t}$).
Each period, only a fraction $1 - \psi$ of retail firms is allowed to reset their price, while the $\psi$ remaining firms are not allowed to do so, like in Calvo (1983) and Yun (1996). The $\psi$ remaining firms are only allowed to index prices by the rate $\pi_{it}$. The firms allowed to adjust prices are randomly selected each period. Once selected, they set prices so as to maximize expected current and future profits, using the nominal stochastic discount factor $\beta \psi$:

$$\max_{P_{f,t}} E_t \left\{ \sum_{s=0}^{\infty} (\beta\psi)^s \Lambda_{t,t+s} (P_{f,t}/P_{t+s}) \left[ P_{f,t} \left( \Pi_{j=1}^{j=s} \theta_{t+j} \right) - P_{t+s}^m \right] y_{f,t+s} \right\} \Rightarrow$$

$$\max_{P_{f,t}} E_t \left\{ \sum_{s=0}^{\infty} (\beta\psi)^s \Lambda_{t,t+s} P_t \left[ \frac{P_{f,t}}{P_{t+s}} \left( \Pi_{j=1}^{j=s} \theta_{t+j} \right) - mc_{t+s} \right] y_{f,t+s} \right\},$$

where $mc_{t+s}$ are the real marginal costs of the retail goods firm in period $t + s$. The demand function $y_{f,t+s}$ is given by $y_{f,t+s} = \left( \Pi_{j=1}^{j=s} \theta_{t+j} \right) P_{f,t}/P_{t+s}^{-\epsilon} y_{t+s}$, where $y_{t+s}$ is the output of the final goods producing firms, and $P_{t+s}$ the price of the final good in period $t + s$. We can rewrite the optimization problem by filling in the demand schedule to obtain:

$$\max_{P_{f,t}} E_t \left\{ \sum_{s=0}^{\infty} (\beta\psi)^s \Lambda_{t,t+s} P_t \left[ \frac{P_{f,t}}{P_{t+s}} \left( \Pi_{j=1}^{j=s} \theta_{t+j} \right) - mc_{t+s} \right] \left( \frac{P_{f,t}}{P_{t+s}} \left( \Pi_{j=1}^{j=s} \theta_{t+j} \right) \right)^{-\epsilon} y_{t+s} \right\} \Rightarrow$$

$$\max_{P_{f,t}} E_t \left\{ \sum_{s=0}^{\infty} (\beta\psi)^s \Lambda_{t,t+s} \left[ \frac{P_{f,t}}{P_{t+s}} \left( \Pi_{j=1}^{j=s} \theta_{t+j} \right) \right]^{1-\epsilon} y_{t+s} - \sum_{s=0}^{\infty} (\beta\psi)^s \Lambda_{t,t+s} P_t mc_{t+s} \left[ \frac{P_{f,t}}{P_{t+s}} \left( \Pi_{j=1}^{j=s} \theta_{t+j} \right) \right]^{-\epsilon} y_{t+s} \right\}.$$

Differentiation with respect to $P_{f,t}$ and using symmetry then yields the new retail price $P_{t}new$:

$$0 = E_t \left\{ \sum_{s=0}^{\infty} (\beta\psi)^s \Lambda_{t,t+s} P_t \left[ \frac{P_{f,t}}{P_{t+s}} \left( \Pi_{j=1}^{j=s} \theta_{t+j} \right) \right]^{1-\epsilon} (1 - \epsilon) \left( \frac{1}{P_{f,t}} \right) y_{t+s} \right\}$$

$$- E_t \left\{ \sum_{s=0}^{\infty} (\beta\psi)^s \Lambda_{t,t+s} P_t \left[ \frac{P_{f,t}}{P_{t+s}} \left( \Pi_{j=1}^{j=s} \theta_{t+j} \right) \right]^{-\epsilon} (\epsilon) \left( \frac{1}{P_{f,t}} \right) mc_{t+s} y_{t+s} \right\}$$

$$\Rightarrow (\epsilon - 1) (P_{f,t})^{-\epsilon} P_t E_t \sum_{s=0}^{\infty} (\beta\psi)^s \Lambda_{t,t+s} \left( \Pi_{j=1}^{j=s} \theta_{t+j} \right)^{1-\epsilon} (P_{t+s})^{\epsilon-1} y_{t+s}$$

$$= \epsilon P_t (P_{f,t})^{-\epsilon-1} E_t \left[ \sum_{s=0}^{\infty} (\beta\psi)^s \Lambda_{t,t+s} \left( \Pi_{j=1}^{j=s} \theta_{t+j} \right)^{1-\epsilon} (P_{t+s})^{\epsilon} mc_{t+s} y_{t+s} \right],$$

$$\Rightarrow \frac{P_{f,t}}{P_{t}} = \left( \frac{\epsilon}{\epsilon - 1} \right) \frac{E_t \left[ \sum_{s=0}^{\infty} (\beta\psi)^s \Lambda_{t,t+s} \left( \Pi_{j=1}^{j=s} \theta_{t+j} \right)^{1-\epsilon} (P_{t+s})^{\epsilon} mc_{t+s} y_{t+s} \right]}{E_t \left[ \sum_{s=0}^{\infty} (\beta\psi)^s \Lambda_{t,t+s} \left( \Pi_{j=1}^{j=s} \theta_{t+j} \right)^{1-\epsilon} (P_{t+s})^{\epsilon-1} y_{t+s} \right]}.$$

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Hence we get the following expression for the new retail price:

\[
\frac{P^{new}}{P_t} = \left( \frac{\epsilon}{\epsilon - 1} \right) \frac{E_t \sum_{s=0}^{\infty} (\beta \psi)^s \Lambda_{t,t+s} \left( \prod_{j=1}^{s} \pi^{adj}_{t+j} \right)^{\epsilon} \left( \prod_{j=1}^{s} \pi_{t+s} \right)^{\epsilon} m_{c+t+s,y_{t+s}}}{E_t \sum_{s=0}^{\infty} (\beta \psi)^s \Lambda_{t,t+s} \left( \prod_{j=1}^{s} \pi^{adj}_{t+j} \right)^{1-\epsilon} \left( \prod_{j=1}^{s} \pi_{t+s} \right)^{1-\epsilon} y_{t+s}}
\]

Defining the relative price of the firms that are allowed to reset their prices as \( \pi^{new}_t = \frac{P^{new}_t}{P_t} \) and gross inflation on final goods as \( \pi_t = \frac{P_t}{P_{t-1}} - 1 \), we can rewrite the first order condition:

\[
\pi^{new}_t = \frac{\epsilon}{\epsilon - 1} \Xi_{1,t}, \quad (10)
\]

\[
\Xi_{1,t} = \lambda_t m c_t y_t + \beta \psi E_t \left[ \left( \pi^{adj}_{t+1} \right)^{-\epsilon} \left( \pi_{t+1} \right)^{\epsilon} \Xi_{1,t+1} \right], \quad (11)
\]

\[
\Xi_{2,t} = \lambda_t y_t + \beta \psi E_t \left[ \left( \pi^{adj}_{t+1} \right)^{1-\epsilon} \left( \pi_{t+1} \right)^{-1} \Xi_{2,t+1} \right]. \quad (12)
\]

The aggregate producer’s price level equals:

\[
(P_t)^{1-\epsilon} = (1 - \psi) (P^{new}_t)^{1-\epsilon} + \psi \left( P_{t-1} \pi^{adj}_t \right)^{1-\epsilon}. \quad (13)
\]

Dividing by \( (P_t)^{1-\epsilon} \) yields the following law of motion:

\[
(1 - \psi) (\pi^{new}_t)^{1-\epsilon} + \psi \left( \pi^{adj}_t \right)^{1-\epsilon} \left( \pi_t \right)^{-1} = 1. \quad (14)
\]

### 1.3.3 Final Goods Producers

Final goods firms purchase intermediate goods which have been repackaged by the retail firms in order to produce the final good. The technology that is applied in producing the final good is given by \( y_f^{(e-1)/\epsilon} = \int_0^1 y_f^{(e-1)/\epsilon} df \), where \( y_{f,t} \) is the output of the retail firm indexed by \( f \). \( \epsilon \) is the elasticity of substitution between the intermediate goods purchased from the different retail firms. The final goods firms face perfect competition, and therefore take prices as given. Thus they maximize profits by choosing \( y_{f,t} \) such that \( P_t y_t = \int_0^1 P_{f,t} y_{f,t} df \) is maximized. Taking the first order conditions with respect to \( y_{f,t} \), gives the demand function of the final goods producers for the retail goods. Substitution of the demand function into the technology constraint gives the relation between the price level of the final goods and the price level of the individual retail firms:

\[
y_{f,t} = (P_{f,t}/P_t)^{-\epsilon} y_t, \\
P_t^{1-\epsilon} = \int_0^1 P_{f,t}^{1-\epsilon} df.
\]
1.3.4 Aggregation

Substituting $y_{i,t} = y_t \left( P_{f,t} / P_t \right)^{-\epsilon}$ into the factor demands derived earlier yields:

$$h_{i,t} = (1 - \alpha) m_t y_{i,t} / w_t, \quad k_{i,t-1} = \alpha m_t y_{i,t} / [q_t^k (1 + r_t^k) - q_t^k (1 - \delta) \xi_t].$$

Aggregation over all firms $i$ gives us aggregate labor and capital:

$$h_t = (1 - \alpha) m_t y_t D_t / w_t, \quad k_{t-1} = \alpha m_t y_t D_t / [q_t^k (1 + r_t^k) - q_t^k (1 - \delta) \xi_t],$$

where $D_t = \int_0^1 \left( P_{f,t} / P_t \right)^{-\epsilon} df$ denotes the price dispersion. It is given by the following recursive form:

$$D_t = (1 - \psi) (\pi_t^*)^{-\epsilon} + \psi \pi_t^* \left( \pi_t^{adj} \right)^{-\epsilon} D_{t-1}. \quad (15)$$

The aggregate capital-labor ratio is equal to the individual capital-labor ratio:

$$k_{t-1} / h_t = \alpha (1 - \alpha)^{-1} w_t / [q_t^k (1 + r_t^k) - q_t^k (1 - \delta) \xi_t] = k_{i,t-1} / h_{i,t}. \quad (16)$$

Now calculate aggregate supply by aggregating $y_{i,t} = a_t (\xi_t k_{i,t-1})^\alpha h_{i,t}^{1-\alpha}$:

$$\int_0^1 a_t (\xi_t k_{i,t-1})^\alpha h_{i,t}^{1-\alpha} \xi_t = a_t \xi_t^\alpha \left( \frac{k_{t-1}}{h_t} \right)^\alpha \int_0^1 h_{i,t} \xi_t d\xi_t = a_t (\xi_t k_{t-1})^\alpha h_t^{1-\alpha},$$

while aggregation over $y_{i,t}$ gives:

$$\int_0^1 y_{i,t} df = y_t \int_0^1 \left( P_{f,t} / P_t \right)^{-\epsilon} df = y_t D_t.$$

So we get the following relation for aggregate supply $y_t$:

$$y_t D_t = a_t (\xi_t k_{t-1})^\alpha h_t^{1-\alpha}. \quad (17)$$

2 Calibration strategies for the default function

In this section we will write down the 2 calibration strategies regarding the sovereign debt in the current paper, since other parts of the model are straightforward to calibrate. The steady state value of a variable $x_t$ is denoted by $\bar{x}_t$. We have the following 2 equations from the financial intermediaries’ problem, from which we can derive the steady state return on bonds ex-post a
possible default.

$$\nu^k_t = E_t \left[ \Omega_{t+1} \left( r^k_{t+1} - r^d_{t+1} \right) \right],$$
$$\nu^b_t = \left( \frac{\lambda^b_t}{\lambda^k_t} \right) \nu^k_t = E_t \left[ \Omega_{t+1} \left( r^{b^*} - r^d_{t+1} \right) \right],$$
$$\implies E_t \left[ \Omega_{t+1} \left( r^{b^*} - r^d_{t+1} \right) \right] = \left( \frac{\lambda^b_t}{\lambda^k_t} \right) E_t \left[ \Omega_{t+1} \left( r^k - r^d_{t+1} \right) \right].$$

From these equations it is clear that $\bar{r}^b_t = \bar{r}^d_t + \left( \frac{\lambda^b_t}{\lambda^k_t} \right) (\bar{r}^k_t - \bar{r}^d_t)$. Now we have the following equation for the maximum level of debt:

$$b^{max}_t = \tilde{b} + \frac{E_t \left[ \tau^{max}_t \right] - \bar{r}}{\lambda^b_t}, \quad (18)$$

The government budget constraint in case of no default by the government is equal to:

$$q^b_t \tilde{b} + \tau_t + \tilde{n}^q_t = g_t + n^q_t + (r^c + \rho q^b_t) b_{t-1},$$
$$= g_t + n^q_t + (r^c + \rho q^b_t) \frac{q^b_{t-1}}{q^b_{t-1}} b_{t-1} \implies$$
$$q^b_t \tilde{b} + \tau_t + \tilde{n}^q_t = g_t + n^q_t + (1 + r^b_t) q^b_{t-1} b_{t-1}. \quad (19)$$

The mapping from the number of no default bonds to the actual number of bonds is given by:

$$b_t = b^{max}_t - \max \left( b^{max}_t - \tilde{b} , 0 \right) \approx b^{max}_t - put \left( b^{max}_t , \tilde{b} \right). \quad (20)$$

The actual number of government bonds is given by:

$$q^b_t \tilde{b} + \tau_t + \tilde{n}^q_t = g_t + n^q_t + (1 - \Delta_t) \left( r^c + \rho q^b_t \right) b_{t-1},$$
$$= g_t + n^q_t + (1 - \Delta_t) \frac{r^c + \rho q^b_t}{q^b_{t-1}} q^b_{t-1} b_{t-1} \implies$$
$$q^b_t \tilde{b} + \tau_t + \tilde{n}^q_t = g_t + n^q_t + (1 - \Delta_t) \left( r^b_t \right) q^b_{t-1} b_{t-1} \implies$$
$$q^b_t \tilde{b} + \tau_t + \tilde{n}^q_t = g_t + n^q_t + (1 + r^b_t) q^b_{t-1} b_{t-1}. \quad (21)$$

together with the ex-post default return on bonds:

$$1 + r^{b^*}_t = (1 - \Delta_t) \left( 1 + r^b_t \right), \quad (22)$$

and the return on bonds before default:

$$1 + r^b_t = \frac{r^c + \rho q^b_t}{q^b_{t-1}}. \quad (23)$$
Throughout the paper we assume that the financial intermediaries do not receive any support from the government in the steady state, nor are they paying back support in the steady state, i.e. $\tilde{n}_g = \tilde{n}_g = 0$. Finally, we have the option pricing formulas:

\[
\text{put}_t = b_{t}^{max} e^{-rT} \Phi(-d_{2,t}) - b_t \Phi(-d_{1,t}), \quad (24) \\
d_{1,t} = \log \left( \frac{\tilde{b}_t}{b_t^{max}} \right) + \left( r + \frac{\sigma^2}{2} \right) T \frac{1}{\sigma \sqrt{T}}, \quad (25) \\
d_{2,t} = \log \left( \frac{\tilde{b}_t}{b_t^{max}} \right) + \left( r - \frac{\sigma^2}{2} \right) T \frac{1}{\sigma \sqrt{T}}. \quad (26)
\]

### 2.1 Calibration strategy 1

The first strategy targets $\bar{q} b$, $\bar{q} b^{max}$ and $\bar{\Delta}$, for which we take 53.2%, respectively 70% of annual GDP and $\bar{\Delta} = 0.0025$. Since we know $\bar{q} b$, the steady state return on bonds after default $\bar{r}_b$ and $\bar{g}$ (which is calibrated to be 17.8% of steady state output), we can find the steady state level of taxes from (21):

\[
\bar{q} b \bar{\tilde{b}} + \bar{\tau} = \bar{g} + (1 + \bar{r}_b) \bar{q} b \Rightarrow \\
\bar{\tau} = \bar{g} + \bar{r}_b \bar{q} b.
\]

Since we know the steady state default fraction $\bar{\Delta}$ and the ex-post return on bonds, we can calculate $\bar{r}_b$ through (22):

\[
1 + \bar{r}_b = (1 - \bar{\Delta}) (1 + \bar{r}_b) \Rightarrow \\
\bar{r}_b = \frac{1 + \bar{r}_b}{1 - \bar{\Delta}} - 1.
\]

Since we know $r_c$, we can find the steady state bond price through (23):

\[
1 + \bar{r}_b = \frac{r_c + \rho \bar{q} b}{\bar{q} b} \Rightarrow \bar{q} b (1 + \bar{r}_b) = r_c + \rho \bar{q} b \Rightarrow \\
\bar{q} b = \frac{r_c}{1 + \bar{r}_b - \rho}.
\]

Now that the steady state bond price is known, we can find the steady state number of bonds and the maximum number of bonds $\bar{b}$ and $\bar{b}^{max}$. Since we know the return on bonds $\bar{r}_b$, we can find the number of bonds if the government does not default:

\[
\bar{q} b \bar{\tilde{b}} + \bar{\tau} = \bar{g} + (1 + \bar{r}_b) \bar{q} b \bar{b}.
\]

Now that we have found steady state number of bonds $\bar{b}$, the maximum number of bonds in the steady state $\bar{b}^{max}$, and the steady state number of bonds in case the government does not default
With these 3 numbers, and the requirement that the derivative of the put option $-\Phi(-d_1, t)$ is equal to $-0.99$ (otherwise the default probability goes down when debt increases, which is counterintuitive), we can find the variables $r, \sigma$ and $T$ from the option pricing formulas.

### 2.2 Calibration strategy 2

The second strategy targets $\tilde{q}_b \bar{b}$ and $\tilde{q}_b \bar{b}_{\text{max}}$, and takes the option pricing parameters $r, \sigma$ and $T$ from calibration strategy 1 as given. We calibrate $\tilde{q}_b \bar{b}$ to be equal to 53.2% of annual GDP, while $\tilde{q}_b \bar{b}_{\text{max}}$ is equal to 60% of annual GDP. Since $\bar{g}$ is also known, we can find the steady state level of taxes from (21):

$$\bar{q}_b \bar{b} + \bar{\tau} = \bar{g} + (1 + \bar{r}_b) \tilde{q}_b \bar{b} \implies \bar{\tau} = \bar{g} + \bar{r}_b \tilde{q}_b \bar{b}.$$

Since we know $\tilde{q}_b \bar{b}$ and $\tilde{q}_b \bar{b}_{\text{max}}$, we can divide the two to find the ratio $\bar{b}_{\text{max}} / \bar{b}$. Now we look at the option pricing formulas, and remember that the parameters $r, \sigma$ and $T$ are given. We rewrite the put option in the following way:

$$b_t = b_{t}^{\text{max}} - \text{put}_t = b_{t}^{\text{max}} - \left\{ b_{t}^{\text{max}} e^{-rT} \Phi(-d_2, t) - \tilde{b}_t \Phi(-d_1, t) \right\}$$

$$= b_t^{\text{max}} - b_t^{\text{max}} e^{-rT} \Phi(-d_2, t) + \tilde{b}_t \Phi(-d_1, t).$$

Division by $b_t$ gives the following expression:

$$1 = \frac{b_{t}^{\text{max}}}{b_t} - \frac{b_{t}^{\text{max}}}{b_t} e^{-rT} \Phi(-d_2, t) + \frac{\tilde{b}_t}{b_t} \Phi(-d_1, t). \quad (27)$$

Now we look at the formula for $d_{1,t}$:

$$d_{1,t} = \frac{\log \left( \frac{\tilde{b}_t/b_{t}^{\text{max}}}{b_t} \right) + \left( r + \frac{\sigma^2}{2} \right) T}{\sigma \sqrt{T}},$$

$$= \frac{\log \left( \frac{\tilde{b}_t}{b_t} \frac{b_{t}^{\text{max}}}{b_t} \right) + \left( r + \frac{\sigma^2}{2} \right) T}{\sigma \sqrt{T}} = f \left( \frac{\tilde{b}_t}{b_t}, \frac{b_{t}^{\text{max}}}{b_t} \right). \quad (28)$$

Similarly we find for $d_{2,t}$:

$$d_{2,t} = \frac{\log \left( \frac{\tilde{b}_t/b_{t}^{\text{max}}}{b_t} \right) + \left( r - \frac{\sigma^2}{2} \right) T}{\sigma \sqrt{T}},$$

$$= \frac{\log \left( \frac{\tilde{b}_t}{b_t} \frac{b_{t}^{\text{max}}}{b_t} \right) + \left( r - \frac{\sigma^2}{2} \right) T}{\sigma \sqrt{T}} = g \left( \frac{\tilde{b}_t}{b_t}, \frac{b_{t}^{\text{max}}}{b_t} \right). \quad (29)$$
Hence we see that equations (27), (28) and (29) only depend on the ratios $b_{t}^{max}/b_t$ and $\tilde{b}_t/b_t$. We know the first ratio in steady state, so we can solve for the steady state ratio $\bar{\tilde{b}}/\bar{b}$. We also see that regarding the default function, it does not matter whether we calibrate on the number of bonds $b_t$ or the value of government liabilities $q^t b_t$, since the bond ratios are the variables that matter. Now we can find $\bar{\tilde{b}}/\bar{b}$, and hence we find $\bar{q}_b \tilde{b} = \bar{q}_b b \left( \tilde{b}/\bar{b} \right)$. Since we know $\bar{q}_b \tilde{b}$, we can find $\bar{r}_b$ from (19):

$$\bar{q}_b \tilde{b} + \bar{\tilde{r}} = \bar{g} + (1 + \bar{r}_b) \bar{q}_b \tilde{b} \implies \bar{r}_b = \frac{\bar{q}_b \tilde{b} + \bar{\tilde{r}} - \bar{g}}{\bar{q}_b \tilde{b}} - 1.$$ 

Now we can find the steady state bond price from (23):

$$1 + \bar{r}_b = \frac{r_c + \rho \bar{q}_b}{\bar{q}_b} \implies \bar{q}_b (1 + \bar{r}_b) = r_c + \rho \bar{q}_b \implies \bar{q}_b = \frac{1 + \bar{r}_b - \rho}{r_c}.$$ 

after which we know $\bar{b}$, $b_{max}^{\text{max}}$ and $\tilde{b}$. From (22) we can find the steady state default probability:

$$1 + \bar{r}_{bs} = \left( 1 - \bar{\Delta} \right) (1 + \bar{r}_b) \implies \bar{\Delta} = 1 - \frac{1 + \bar{r}_{bs}}{1 + \bar{r}_b}.$$

### 3 Robustness checks

In this section we investigate whether the results we obtain are robust under variation of some key parameters. This is obviously important, as we need to make sure that our results are not driven by particular parameter values. The credit spread between the return on assets and deposits is a key variable. Two variables that can potentially affect this credit spread are the interest rate smoothing parameter $\rho$, and the price-indexation parameter $\gamma_P$. A third ratio that we vary is the ratio of the diversion rate of government bonds over the diversion rate of private loans. We investigate whether the key mechanism that explains why the announcement of a recapitalization of the Spanish banking system by the Spanish government failed, namely the negative amplification between weakly capitalized banks with large holdings of risky sovereign debt and weak government finances, is still present under alternative parameter choices for $\rho$, $\gamma_P$ and $\lambda_0/\lambda_t$ respectively. We do so in Figures 1, 2 and 3 respectively, and find that our results are robust under these alternative parameter specifications.
References


Eurostat (2014). Quarterly government debt.


Financial crisis, no recap vs. debt financed recap, $\rho_r = 0.4$

(a) Diversion rate private loans vs. Output
(b) Intermediary Net Worth vs. Credit Spread $E[r_k - r_d]$

Figure 1: Impulse response functions for the model runs including sovereign default risk and long term debt with a delayed recapitalization by the government (red, slotted) of 12 per cent of quarterly steady state output occurring eight quarters after the capital quality shock and for the simulations without additional government policy (blue, solid).
Figure 2: Impulse response functions for the model runs including sovereign default risk and long term debt with a delayed recapitalization by the government (red, slotted) of 12 per cent of quarterly steady state output occurring eight quarters after the capital quality shock and for the simulations without additional government policy (blue, solid).
Financial crisis, no recap vs. debt financed recap, $\bar{\lambda}_b/\bar{\lambda}_k = 0.25$

(a)

(b)

Figure 3: Impulse response functions for the model runs including sovereign default risk and long term debt with a delayed recapitalization by the government (red, slotted) of 12 per cent of quarterly steady state output occurring eight quarters after the capital quality shock and for the simulations without additional government policy (blue, solid).