Proof of Lemma 1

Consider a projection matrix satisfying $H_p X_1 = 0$, $H_p X_2 = X_2$, and $H_p Z_2 = Z_2$ in the balanced design. We first show that all the elements in $y_{rr}$ have the same expectation under null hypothesis. By pre-multiplying Equation 3 by $H_p$ we get

$$y_{rr} = H_p y = X_2 \beta_2 + Z_1 \gamma_1 + Z_2 \gamma_2 + \epsilon_{rr},$$

where $Z_1 \gamma_1 = H_p Z_1$ and $\epsilon_{rr} = H_p \epsilon$. Note that the first property of $H_p$ guarantees that the fixed effects of $\beta_1$ is wiped away in this model. Now, the random effects and error terms $\gamma_1$, $\gamma_2$ and $\epsilon_{rr}$ have all a zero expectation. Moreover, under the null hypothesis 2, $\beta_2 = 0$. This implies that the vector $y_{rr}$ has zero expectation under the null hypothesis.

It remains to show that $F_{rr} = F_{ss}$. To this end, we use the fact that $H_p X_2 = X_2$, which implies

$$y_{rr} H_{[X_2]} y_{rr} = (y' H_p)(X_2(X_2'X_2)^{-1}X_2')(H_p y) = y'(H_p X_2)(X_2'X_2)^{-1}(X_2' H_p) y$$

$$= y' X_2(X_2'X_2)^{-1}X_2' y = y' H_{[X_2]} y.$$

Similarly, we have $y_{rr} H_{[Z_2]} y_{rr} = y' H_{[Z_2]} y$. Therefore,

$$F_{rr} = \frac{y_{rr}' H_{[X_2]} y_{rr} / p_2}{y_{rr}' H_{[Z_2]} y_{rr} / q_2} = \frac{y' H_{[X_2]} y / p_2}{y' H_{[Z_2]} y / q_2} = F_{ss}.$$

Proof of Theorem 1

The proof is similar to the Lemma 1, but we should mention that in this case also

$$y_{rr} H_{[X_{2r}]} y_{rr} = y'(I - H_{[X_1]}) H_{[X_{2r}]} (I - H_{[X_1]}) y = y' H_{[X_{2r}]} y.$$

Proof of Theorem 2

Due to lemma 1, and the fact that we introduced modified residuals only for balanced designs, it merely remains to prove that $F_{mr} = F_{rr}$. First note that

$$y_{mr}' X_{mr} = y_{rr} V V' X_2 = \lambda y_{rr}(H_{nd}) X_2 = \lambda y_{rr}' X_2,$$

$$X_{mr}' X_{mr} = X_2' V V' X_2 = \lambda X_2' (H_{nd}) X_2 = \lambda X_2' X_2,$$

and

$$y_{mr} y_{mr} = y_{rr}' V V' y_{rr} = \lambda y_{rr}(H_{nd}) y_{rr} = \lambda y_{rr}' y_{rr}.$$
where we have used that $V^TV' = \lambda H_{nd}$ and based on Theorem 3 and 5, which shows the eigen-values of numerator and denominator are the same, and we have

$$F_{mr} = \frac{y_{mr}'H_{|X_{mr}|}y_{mr}}{y_{mr}'H_{|Z_{mr}|}y_{mr}} = \frac{y_{mr}'(X_{mr}'X_{mr})^{-1}X_{mir}'y_{mr}}{y_{mr}'(Z_{mr}'Z_{mr})^{-1}Z_{mir}'y_{mr}}$$

$$= \frac{\lambda y_{rr}'X_2(\frac{1}{\lambda})(X_2'X_2)^{-1}X_2^2y_{rr}}{\lambda y_{rr}'Z_2(\frac{1}{\lambda})(Z_2'Z_2)^{-1}Z_2^2y_{rr}}$$

$$= \frac{y_{rr}'X_2(X_2'X_2)^{-1}X_2^2y_{rr}}{y_{rr}'Z_2(Z_2'Z_2)^{-1}Z_2^2y_{rr}}$$

$$= \frac{y'_{H_{|X_2|}y}}{y'_{H_{|Z_2|}y}}$$

\[\blacksquare\]

Proof of Theorem 3

The following definitions are required in the proofs of Theorem 3, 4 and 5. As we need Helmert contrasts in order to parametrize the designs matrices, we recall here the construction of the Helmert orthonormal contrasts (see e.g. Sahai and Ageel 2000).

**Definition 1** For an integer $k$ and $i = 1, \ldots, k-1$, define $q_{ik}^k$ to be a vector of length $k$, with

$$q_{ik}^k(j) = \begin{cases} \frac{i}{\sqrt{i(i+1)}} & \text{if } j \leq i \\ \frac{-i}{\sqrt{i(i+1)}} & \text{if } j = i + 1 \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, define $Q_k$ to be a matrix of size $k \times (k-1)$ whose columns are $q_{ik}^k$’s,

$$Q_k = [q_{1k}^1, q_{2k}^2, \ldots, q_{k-1}^{k-1}] .$$

**Lemma 3** The set of vectors $\{q_{1k}^1, q_{2k}^2, \ldots, q_{k-1}^{k-1}\}$ are $k$ orthonormal vectors, i.e.,

$$q_{ik}^k q_{jk}^k = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, $Q_k^tQ_k = I_{k-1}$. Moreover, we have $Q_k^tQ_k' = \Delta_k = I_k - I_{k \times k} / k$. Note that the above is true for any set of $k-1$ orthonormal contrasts.

The proof of this lemma is omitted since it straightforward. The central part to prove the Theorem is to show that $\Sigma_{rr} = \lambda H_{nd}$. We first construct the matrix $H_{nd}$. Based on model (20) and on definition 1, for repeated measures ANOVA with one within-subject factor, we have that

$$X_q = I_{s \times 1} \otimes Q_b \otimes I_{n \times 1}, \quad X_z = Q_s \otimes I_{b \times 1} \otimes I_{n \times 1}.$$  \[26\]
Therefore, we can construct $X_{\eta\pi}$ as

$$X_{\eta\pi} = Q_s \otimes Q_b \otimes I_{n \times 1}.$$ 

Note that

$$X'_{\eta}X_{\eta} = (I_{1 \times s} \otimes Q'_b \otimes I_{1 \times n}) (I_{s \times 1} \otimes Q_b \otimes I_{n \times 1})$$

$$= (s I_{1 \times 1}) \otimes (Q'_b Q_b) \otimes (n I_{1 \times 1})$$

$$= (s I_{1 \times 1}) \otimes I_{s-1} \otimes (n I_{1 \times 1})$$

$$= snI_{b-1}.$$ 

Hence, for $H_{[X_{\eta}]} = X_{\eta}(X'_{\eta}X_{\eta})^{-1}X'_{\eta}$, we have

$$H_{[X_{\eta}]} = X_{\eta}(X'_{\eta}X_{\eta})^{-1}X'_{\eta}$$

$$= \frac{1}{sn} X_{\eta}X'_{\eta}$$

$$= \frac{1}{sn} (I_{s \times 1} \otimes Q_b \otimes I_{1 \times n}) (I_{s \times 1} \otimes Q'_b \otimes I_{1 \times n})$$

$$= \frac{1}{sn} (I_{s \times s} \otimes (Q_b Q'_b) \otimes I_{n \times n})$$

$$= \frac{1}{sn} (I_{s \times s} \otimes \Delta_b \otimes I_{n \times n}).$$ 

Similarly, for $X_{\eta\pi}$, we have $X'_{\eta\pi}X_{\eta\pi} = (Q'_s \otimes Q'_b \otimes I_{1 \times n}) (Q_s \otimes Q_b \otimes I_{n \times 1}) = nI_{(s-1)(b-1)}$ and $H_{[X_{\eta\pi}]} = X_{\eta\pi}(X'_{\eta\pi}X_{\eta\pi})^{-1}X'_{\eta\pi} = \frac{1}{n} (\Delta_s \otimes \Delta_b \otimes I_{n \times n})$. Therefore, we can write

$$H_{nd} = H_{[X_{\eta}]} + H_{[X_{\eta\pi}]}$$

$$= \frac{1}{sn} (I_{s \times s} \otimes \Delta_b \otimes I_{n \times n}) + \frac{1}{n} (\Delta_s \otimes \Delta_b \otimes I_{n \times n})$$

$$= \frac{1}{sn} \left( \frac{1}{s} I_{s \times s} + \Delta_s \right) \otimes \Delta_b \otimes I_{n \times n}$$

$$= \frac{1}{n} I_{s \times s} \otimes \Delta_b \otimes I_{n \times n}.$$ 

In the following, we derive $S_{rr}$ in a simple way, based on (22):

$$S_{rr} = H_{nd} \left[ \sigma^2_{X_{\eta\pi}X_{\eta\pi}} + X'_{\eta\pi} \text{cov}(\eta\pi)X_{\eta\pi} + \sigma^2 I_N \right] H_{nd}$$

$$= \frac{1}{n^2} \left( I_s \otimes \Delta_b \otimes I_{n \times n} \right) \left[ \sigma^2_{X_{\eta\pi}} (I_s \otimes \Delta_b \otimes I_{n \times n}) + \sigma^2 (I_s \otimes I_b \otimes I_n) \right] (I_s \otimes \Delta_b \otimes I_{n \times n})$$

$$= \frac{1}{n^2} \left[ \sigma^2 (I_s \otimes \Delta_b \otimes I_{n \times n}) + \sigma^2 (I_s \otimes \Delta_b \otimes n I_{n \times n}) \right]$$

$$= \frac{1}{n^2} \left[ \sigma^2 (I_s \otimes \Delta_b \otimes n^2 I_{n \times n}) + \sigma^2 (I_s \otimes \Delta_b \otimes n I_{n \times n}) \right]$$

$$= \frac{1}{n} (\sigma^2 + \sigma^2) (I_s \otimes \Delta_b \otimes I_{n \times n})$$

$$= \frac{1}{n} (I_s \otimes \Delta_b \otimes I_{n \times n}),$$
where we used the facts that $H_{nd}X_\eta\eta' = 0$, $\Delta_b\Delta_b = \Delta_b$ and $\mathbb{I}_{n \times n} \mathbb{I}_{n \times n} = n \mathbb{I}_{n \times n}$. According to the definition of $H_{nd}$ and $\Sigma_{rr}$ we have shown that $\Sigma_{rr} = \lambda H_{nd}$, where, in the repeated measures ANOVA with one within-subject factor, $\lambda$ is $(n\sigma_\omega^2 + \sigma_\epsilon^2)$, which proves the first assertion.

To show the second assertion of Theorem 3 we use the fact that $H_{nd}$ is an idempotent matrix, i.e. it has only two eigen-values 0 and 1. This last eigen-value has multiplicity equal to the rank of $H_{nd}$ or number of (linearly independent) projection columns, which is equal to the number of columns of $X_\eta$ and $X_\eta\pi$. So the multiplicity is $(b-1) + (b-1)(s-1)$, i.e. $(b-1)s$. Then the only non-zero eigen-value of $\Sigma_{rr}$ is $\lambda = (n\sigma_\omega^2 + \sigma_\epsilon^2)$, where $\sigma_\omega^2 = \frac{b}{b-1}\sigma_\eta^2$ with the above multiplicity.

Proof of Lemma 2

From the definition of $\Sigma_{rr}$, the covariance matrix of $y_{rr}$, and the fact that $E(y_{rr}) = 0$, we have

$$E[\nu_1] = \frac{1}{N} \sum_{ijk} E[y_{rr}(ijk)^2] = \frac{1}{N} \sum_{ijk} (\Sigma_{rr})(ijk),(ijk),$$

where, with slightly abuse of notation, we have used $(ijk)$ as a row (column) index for matrix $\Sigma_{rr}$. In fact, the $(ijk)$-th row (column) is the row (column) corresponding to $y_{ijk}$ is physically located in $[(j-1)bn + (i-1)n + k]$-th row (column) of the matrix. Similarly, we have

$$E[\nu_3] = \frac{1}{n^2b(b-1)s} \sum_{j} \sum_{i \neq i'} \sum_{k,k'} \sum_{j} \sum_{i \neq i'} \sum_{k,k'} E[y_{rr}(ijk)y_{rr}(i'jk')].$$

As we show in the proof of Theorem 3, the covariance matrix $\Sigma_{rr}$ is given by

$$\Sigma_{rr} = \frac{\lambda}{n} (I_s \otimes \Delta_b \otimes \mathbb{I}_{n \times n}).$$

Therefore, $\Sigma_{rr}$ has a block-diagonal form and its elements can be easily computed as

$$(\Sigma_{rr})(ijk),(i'j'k') = \begin{cases} \frac{(b-1)\lambda}{bn} & \text{if } i = i', j = j', k = k', \\ \frac{\lambda}{bn} & \text{if } i \neq i', j = j', \\ 0 & \text{otherwise.} \end{cases}$$

Hence, by replacing the values of the elements of $\Sigma_{rr}$ from (29) in (27) and (28), we get

$$E[\hat{\lambda}] = nE[\nu_1] - nE[\nu_3] = n \frac{(b-1)\lambda}{bn} - n \frac{-\lambda}{bn} = \lambda.$$
Proof of Theorem 4

A similar proof as for one within-subject factor can be written in order to show that for two within-subject factors,

$$\Sigma_{rr} = \lambda H_{nd}. $$

In this model, $\eta$ and $\gamma$ are fixed effects and within-subject factors, and $\pi$ is the random effect due to the subjects. To construct the test for one within-subject factor, say $\eta$, then we can show that $X_\eta = Q_s \otimes I_{b \times 1} \otimes I_{n \times 1}$, $X_\eta = I_{s \times 1} \otimes Q_b \otimes I_{n \times 1}$ and $X_{\eta\pi} = Q_s \otimes Q_b \otimes I_{n \times 1}$. Similarly to the previous proof, we can show that

$$H_{[X_\eta]} = \frac{1}{an} I_s \otimes \Delta_b \otimes I_{n \times n}, \quad H_{[X_{\eta\pi}]} = \frac{1}{an} I_s \otimes \Delta_b \otimes I_{n \times n}. $$

In order to construct $\Sigma_{rr}$, based on Equation (23), at first we have

$$\Sigma_{rr} = H_{nd}(\sigma^2_{\eta\pi} X_\pi^\prime X_{\pi} + X_{\eta\pi}^\prime \text{cov}(\eta\pi)X_{\eta\pi} + X_{\eta\gamma\pi}^\prime \text{cov}(\gamma\pi)X_{\eta\gamma\pi} + X_{\eta\gamma\pi}^\prime \text{cov}(\eta\gamma\pi)X_{\eta\gamma\pi} + \sigma^2_{\pi I_N})H_{nd}$$

It is easy to show that, $H_{nd}$ is orthogonal to the to the spaces spanned by, $X_\pi^\prime$, $X_{\gamma\pi}^\prime$ and $X_{\eta\gamma\pi}^\prime$. For example,

$$H_{nd}(X_{\eta\gamma\pi}^\prime \text{cov}(\eta\gamma\pi)X_{\eta\gamma\pi})H_{nd}$$

$$= \frac{\sigma^2_{\eta\gamma\pi}}{(an)^2} [I_s \otimes \Delta_b \otimes I_{a \times a} \otimes I_{n \times n}](I_s \otimes \Delta_b \otimes \Delta_a \otimes I_{n \times n})(I_s \otimes \Delta_b \otimes I_{a \times a} \otimes I_{n \times n})]$$

$$= \frac{\sigma^2_{\eta\gamma\pi}}{(an)^2} [I_s \otimes \Delta_b \otimes 0 \otimes n^2 I_{n \times n}] = 0$$

The same proof can be done to show that

$$H_{nd}(\sigma^2_{\eta\pi} X_\pi^\prime X_{\pi})H_{nd} = 0, \quad H_{nd}(X_{\gamma\pi}^\prime \text{cov}(\gamma\pi)X_{\gamma\pi})H_{nd} = 0.$$

Then we can see that,

$$\Sigma_{rr} = H_{nd}(X_{\eta\pi}^\prime \text{cov}(\eta\pi)X_{\eta\pi} + \sigma^2_{\pi I_N})H_{nd} = \frac{1}{an} (an\sigma^2_{\omega} + \sigma^2_{\pi})[I_s \otimes \Delta_b \otimes I_{n \times n}]$$

which means that in repeated measures designs with two within-subject factors,

$$\Sigma_{rr} = (an\sigma^2_{\omega} + \sigma^2_{\pi})H_{nd},$$

and that $\lambda = (an\sigma^2_{\omega} + \sigma^2_{\pi})$. Based on the fact that $H_{nd}$ is an idempotent matrix with only 0 and 1 as eigen-values, the multiplicity of the second eigen-value is equal to the rank of $H_{nd}$ or number of (linearly independent) projection columns, which is equal to the number of columns of $X_\eta$ and $X_{\eta\pi}$. We thus see that here again $\Sigma_{rr}$ has a unique non-zero eigen-value $\lambda$ with multiplicity $s(b - 1).$
Proof of Theorem 5

To obtain the eigen-values of \( \Sigma_{rr} \) for the mixed-model designs, for testing the between-subject factor, we first define the corresponding design matrices:

\[
X_\eta = Q_\eta \otimes 1_a \otimes 1_s \otimes 1_b \otimes 1_n, \quad X_\pi = I_a \otimes Q_s \otimes 1_b \otimes 1_n.
\]

According to these definitions we can easily calculate

\[
H_{[X_\eta]} = \frac{1}{sb}(\Delta_\eta \otimes 1_{sb \times sb}), \quad H_{[X_\pi]} = \frac{1}{nb}(I_a \otimes \Delta_\pi \otimes 1_{b \times b}) \otimes 1_n, \quad H_{nd} = \frac{1}{ab} (1_a \otimes I_s \otimes 1_{b \times b} \otimes 1_n).
\]

In order to construct \( \Sigma_{rr} \) for this model, using model (25), we define \( X_\eta^o = I_{as} \otimes 1_{nb \times 1} \) and \( X_\pi^o \text{cov}(\eta \pi) X_\eta^o = \sigma_\pi^2(I_{as} \otimes \Delta_\pi \otimes 1_{a \times n}) \) where \( \Delta_\pi = I_b - \frac{1}{b} I_{b \times b} \) and \( \text{cov}(\eta \pi) = (I_{as} \otimes \Delta_\pi)\sigma_\pi^2 \). Then \( \Sigma_{rr} \) can be written as

\[
\Sigma_{rr} = [H_{nd}(X_\eta^o I_a \sigma_\pi^2 X_\eta^o + X_\pi^o \text{cov}(\eta \pi) X_\eta^o + IN \sigma_\pi^2) H_{nd}]
= (I_{a \times a} \otimes I_{nb})((X_\eta^o I_a \sigma_\pi^2 X_\eta^o) + IN \sigma_\pi^2(I_{a \times a} \otimes I_{nb})]
= \frac{1}{abn} (bn\sigma_\pi^2 + \sigma_\pi^2)(I_{a \times a} \otimes I_s \otimes 1_{b \times b} \otimes 1_n).
\]

Using the fact that \( H_{nd} \) is orthogonal to the space spanned by the columns of \( X_\eta^o \), it can be easily shown that \( X_\eta^o \text{cov}(\eta \pi) X_\eta^o H_{nd} = 0 \). Then we have

\[
\Sigma_{rr} = (bn\sigma_\pi^2 + \sigma_\pi^2) H_{nd},
\]

which implies, \( \Sigma_{rr} \) has only one non-zero eigen-value \( \lambda \) with multiplicity equal to the total number of columns of \( X_\pi \) and \( X_\eta \) which is \( (a - 1) + (s - 1) \).

Proof of Theorem 6

The proof is very similar to the one for Theorem 3. We just mention that for this case

\[
X_\eta = 1_a \otimes I_a \otimes Q_b \otimes 1_n, \quad X_\pi = 1_a \otimes Q_s \otimes Q_b \otimes 1_n.
\]

Therefore, we can show that

\[
H_{[X_\eta]} = \frac{1}{asn} (I_{a \times a} \otimes I_a \otimes \Delta_b \otimes 1_{a \times n}), \quad H_{[X_\pi]} = \frac{1}{na} (I_{a \times a} \otimes \Delta_s \otimes \Delta_b \otimes 1_{a \times n}),
\]

\[
H_{nd} = \frac{1}{an} (I_{a \times a} \otimes I_s \otimes \Delta_b \otimes 1_{a \times n})
\]

and in order to construct the \( \Sigma_{rr} \) and refer to the (25), and it can be easily shown that \( \Sigma_{rr} = (n\sigma_\pi^2 + \sigma_\pi^2) H_{nd} \). We skip the details and refer to the previous proofs, however it is worth mentioning that in this case the total number of non-zero eigenvalues of \( \Sigma_{rr} = (b - 1) + a(b - 1)(s - 1) \), which are the total number of column of matrix \( X_\eta \) and \( X_\pi \).