1. Assume a cusp bifurcation point is present. This implies there will be a fold bifurcation point and hence hysteresis in its surrounding. That is, for some values of parameters \( r, K, m, e, h, \) and hence of compound parameters \( C_1 \) and \( C_2 \), there will be a coexistence equilibrium \([N^*, P^*]\) and values for parameters of the predator encounter rate \( \lambda(P) \) that will correspond to the cusp bifurcation point.

2. Let \( \omega \) be one of the parameters of the predator encounter rate \( \lambda(P) \). Let us observe a hysteresis in \( P^* \) with respect to \( \omega \). Then, in an interval of \( \omega \) covering the S-shaped curve, there will be one, two or three values of \( P^* \) corresponding to predator equilibrium densities. Looking at the S-shaped curve after switching \( x \) and \( y \) axes, it will be (locally) represented by a function \( \omega = G(P^*) \). This function \( G(P^*) \) may be a function of \( P^* \) but also of \( \lambda(P^*) \).

3. For a point \((\omega_f, P_f^*)\) to be a fold bifurcation point, the condition \( G'(P^*) = 0 \) has to hold, which is the condition (8). To this condition we insert our specific form of \( \lambda(P) \) and after some rearrangement we get an equation \( g(\Lambda, \Omega, C_1, C_2) = 0 \), where \( \Lambda = \lambda(P^*) \) and \( \Omega \) is the vector-valued parameter of the predator encounter rate \( \lambda(P) \). In our case of \( \lambda(P) \) given by the formula (3) we get the formula (10). Note that not all of the parameters from \( \Omega \) need to be present in the emerging equation. In our case, \( w \) and \( b \) are present but \( \lambda_0 \) is not.

4. To find a fold bifurcation point, we first fix the part of \( \Omega \) present in the formula (10) (parameters \( w \) and \( b \) in our case). Then we solve this formula for \( \Lambda \) to get \( \Lambda_f \). Once we have it (one or more solutions may exist), we insert it for \( \Lambda \) into the formula \( \Lambda = \lambda(P^*) \) in which \( P^* \) and the part of \( \Omega \) absent in the formula (10) (parameter \( \lambda_0 \) in our case) stay undetermined. To find \( P_f^* \) we use the formula (5): \( P_f^* = C_1(\Lambda_f - C_2)/\Lambda_f^2 \). Eventually, we calculate (or choose and calculate in the case of two or more free parameters) the part of \( \Omega \) absent in the formula (10) from the formula \( \Lambda_f = \lambda(P_f^*) \). In parallel, we use the equation (4) in which we insert \( \Lambda_f \) for \( \lambda(P_f^*) \) to calculate \( N_f^* \).

5. Finding a fold bifurcation point does not by itself ensure existence of a cusp bifurcation point. To have a hysteresis, we need to have at least two solutions of (10). These two solutions merge in the cusp bifurcation point. In our case, the condition (11) must hold and we look for a pair of parameters \( w = w_c \) and \( b = b_c \) that ensures this. For these parameters we calculate the respective
Λ = Λ_c. Then we go on as in the previous point. We insert it into the formula Λ_c = \lambda(P^*)
and determine \( P^*_c \) from the formula (5) and calculate the part of Ω absent in the formula (10), i.e. \( \lambda_0^* \) in our case, from \( \Lambda_c = \lambda(P^*_c) \). Finally, or in parallel, we use the equation (4) to calculate \( N^*_c \). If such a point \([N^*_c, P^*_c]\) exists, we have detected a cusp bifurcation and hence hysteresis in a model with given \( \lambda(P) \).

**Example 1** We will now illustrate these points numerically for the predator encounter rate \( \lambda(P) \) (3) and the parameters \( m = 1, e = 1, r = 3, h = 0.25, \) and \( K = 8 \), which give \( C_1 = er/(e - hm) = 4 \) and \( C_2 = m/[K(e - hm)] = 1/6 \).

First, we find a fold bifurcation point. The formula (10) is now

\[ b\Lambda^2 + 4(1 - w)\Lambda + \frac{2}{3}(2w - 1) = 0 \]

which solves as

\[ \Lambda_{f1,f2} = 2 \frac{w - 1}{b} \pm \frac{\sqrt{4(w - 1)^2 - \frac{2}{3}b(2w - 1)}}{b} \]

Choosing \( b = 2 \) and \( w = 3 \) we now have \( \Lambda_{f1} = 2 + \sqrt{7/3} \) and \( \Lambda_{f2} = 2 - \sqrt{7/3} \). We now need to find \( P^*_f \) that satisfies

\[ P^*_f = \frac{C_1(\Lambda_f - C_2)}{\Lambda_f^2} \]

\( \lambda_0^f \) that satisfies

\[ \Lambda_f = \frac{\lambda_0^f}{(2 + P^*_f)^3} \]

and \( N^*_f \) that satisfies

\[ N^*_f = \frac{KC_2}{\Lambda_f} \]

For \( \Lambda_f = \lambda_{f1} \), we get \( P^*_f = 1.0804, \lambda_0^f = 103.1041, \) and \( N^*_f = 0.378 \). For \( \Lambda_f = \lambda_{f2} \), we get \( P^*_f = 5.4796, \lambda_0^f = 197.7061, \) and \( N^*_f = 2.822 \).

Second, we find a cusp bifurcation point. The formula (11) must hold and we look for \( b \) and \( w \) such that

\[ 4(w - 1)^2 - \frac{2}{3}b(2w - 1) = 0 \]

or

\[ w^2 - 2w \left( 1 + \frac{b}{6} \right) + \left( 1 + \frac{b}{6} \right) = 0 \]
For $b_c = 2$ we get two solutions, $w_c = 2$ and $w_c = 2/3$, the latter of which is not of an interest since we know fold bifurcations cannot occur for $w \leq 1$. To $b_c = 2$ and $w_c = 2$ corresponds $\Lambda_c = 1$. Hence, we look for $P_c^*$ such that

$$P_c^* = \frac{C_1(\Lambda_c - C_2)}{\Lambda_c^2} \Rightarrow P_c^* = 10/3$$

$\lambda_0$ such that

$$\frac{\lambda_0^2}{(2 + P_c^*)^2} = 1 \Rightarrow \lambda_0^2 = (16/3)^2$$

and $N_c^*$ such that

$$N_c^* = \frac{KC_2}{\Lambda_c} \Rightarrow N_c^* = 4/3$$

**Example 2** As a second example, we assume the sigmoidally decreasing predator encounter rate

$$\lambda(P) = \lambda_0 \frac{bw}{bw + Pw}, \; w > 1$$

and will look for a cusp bifurcation point. The formula (8) now implies the following necessary condition for the fold bifurcation to occur:

$$\frac{w}{\lambda_0} \Lambda^2 + \left(1 - w \left(1 + \frac{2C_2}{\lambda_0}\right)\right) \Lambda + C_2(2w - 1) = 0$$

Note that this equation is independent of $C_1$ and hence of the parameter $r$. Note that it is now $\lambda_0$ and $w$ that is in the equation for $\Lambda$, while $b$ is now absent. Since $C_2(2w - 1) > 0$ for $w > 1$ and $w \left(1 + \frac{2C_2}{\lambda_0}\right) > 1$ for $w > 1$ then if this equation has real roots then both will be positive. Assume again the parameters $m = 1$, $e = 1$, $r = 3$, $h = 0.25$, and $K = 8$, which give $C_1 = er/(e - hm) = 4$ and $C_2 = m/[K(e - hm)] = 1/6$.

Choosing $w_c = 2$, we can calculate that the discriminant of the previous equation vanishes for approximately $\lambda_0^{c1} = 2.488$ and $\lambda_0^{c2} = 0.1786$, since

$$\lambda_0^c = \frac{2wC_2}{(w - 1)^2} \left(w \pm \sqrt{2w - 1}\right)$$

so we have two candidates for the cusp point. This results in approximately $\Lambda_{c1} = 0.7887$ and $\Lambda_{c2} = 0.2113$, respectively, since

$$\Lambda_c = \frac{\lambda_0^c(w - 1)}{2w} + C_2$$

The formula

$$\Lambda = \lambda_0 \frac{bw}{bw + (P^*)w}, \; w > 1$$
Figure 1: A bifurcation diagram for the parameters $w$ and $\lambda_0$ for the model (3) with the sigmoidally decreasing predator encounter rate (22), showing the cusp bifurcation (CP). The solid black line is the line of fold bifurcation points (or limit points, LP), the gray line is the line of Hopf bifurcation points (H), and the dashed black line is the line of homoclinic bifurcation points (HL). Other model parameters: $m = 1, e = 1, r = 3, h = 0.25, b = 2.725$ and $K = 8$.

implies

$$b_c = P_c^* \left/ \left( \frac{\lambda_0 - \Lambda_c}{\Lambda_c} \right)^{\frac{1}{w}} \right.$$ 

where

$$P_c^* = \frac{C_1(\Lambda_c - C_2)}{\Lambda_c^2}$$

For $\Lambda_c = \lambda_{c1}$, this results in $P_c^* = 4, b_c = 2.725$, and $N_c^* = 1.6906$. For $\Lambda_c = \lambda_{c2}$, this results in $P_c^* = 4, b_c = -10.1698i$, and $N_c^* = 6.3094$. Note that $b_c$ is now a complex number so that no cusp point corresponds to $\Lambda_c = \lambda_{c2}$. The following plot shows a two-parameter bifurcation diagram that verifies our calculations.

We also show a three dimensional ‘cut’ through the bifurcation diagram plotted in Fig. 1 for $w = 3$. In this plot, one can clearly identify both limit points, the Hopf bifurcation point, as well as two homoclinic bifurcations at which the stable limit cycles (dis)appear. The leftmost point on the equilibrium manifold is a branch point at which predator population ceases to exist.

**Example 3** As the final example, we assume the predator encounter rate of the form

$$\lambda(P) = \frac{\lambda_0}{b + P}$$
Figure 2: Dependence of the non-trivial equilibria and limit cycles for the model (3) with the sigmoidally decreasing predator encounter rate (22) on the parameter $\lambda_0$. The solid black line is the line of stable equilibria, the dotted line marks the saddle points and the dashed line is for unstable nodes or foci. Other model parameters: $m = 1, e = 1, r = 3, h = 0.25, b = 2.725, K = 8$ and $w = 3$. $N$ and $P$ denote prey and predator population density, respectively.
This form of $\lambda(P)$ gives rise to the Beddington-DeAngelis functional response for which we know that no cusp bifurcation point exists. Inserting $w = 1$ into the condition (10) gives

$$\Lambda^2 = -\frac{C_1 C_2}{b}$$

which obviously does not have a real solution. Hence, indeed, no fold bifurcation and therefore no cusp bifurcation can occur for the Beddington-DeAngelis functional response.