Figure 3: Symmetry cells and orbit space. The blue lines describe orbits and the red lines describe symmetry cells $A_1$ and $A_2$ defined by the cross-sections $\kappa_1$ and $\kappa_2$. The black line describes the orbit space $M/G$.

6 Orbit Space

Due to invariance of fields, it is sufficient to know the fields at one point of each orbit. In order to construct a domain for a reduced BVP, one point from each orbit must be chosen such that the resulting whole is a manifold. These domains, or symmetry cells, are submanifolds of the domain of a symmetric BVP. In dimensional reduction, the symmetry cells are lower-dimensional submanifolds. There is no canonical choice of symmetry cell. However, all the symmetry cells are required to be canonically diffeomorphic. Then the set of all orbits can be given a canonical manifold structure that is independent of the choice of symmetry cell [6, p. 109]. With this manifold structure, the set of all orbits is called the orbit space and it is the canonical domain for the reduced BVPs.

Let us next define the symmetry cells and orbit space formally. The set of all orbits of $M$ under $G$ is denoted by $M/G$. There is a natural projection $\pi: M \to M/G$ such that each point of $M$ is mapped to its orbit. $\pi$ induces the quotient topology for $M/G$ from $M$ such that the topology is compatible with the orbits: $U \subset M/G$ is open if and only if its preimage $\pi^{-1}(U)$ is an open set of $M$. This topology makes $\pi$ continuous [9] and together with $\pi$ we can now define the symmetry cells (or $G$-reduced domains as in [6]):

Definition 9. An embedded submanifold-with-boundary $A$ of $M$ is a symmetry cell, if there is a continuous mapping $\kappa: M/G \to M$ called cross-section such that $\kappa(M/G) = A$ holds and $\pi \circ \kappa$ is the identity mapping of $M/G$. $\kappa$ is called cross-section because it maps each orbit to one of its points (see Fig. 3). Each cross-section is a homeomorphism to its range and induces a manifold structure for $M/G$ from the requirement that the cross-section be a homeomorphism from $M/G$ to its range. Finally, in order to make the induced manifold structure independent of the choice of the cross-section and the symmetry cells must be canonically diffeomorphic (an induced domain from symmetry cell $A_1$ to symmetry cell $A_2$ must be a diffeomorphism).

Definition 10. The set of all orbits $M/G$ together with a differentiable manifold structure is the orbit space, if the manifold structure is induced from a symmetry cell and is independent of the choice of the symmetry cell.