Mathematical details on the two-level hierarchy

Taking up multi-level hierarchy in the main manuscript, we assume that the neural field supports a hierarchy of stable heteroclinic sequences that these SHS are given by (possibly infinitely many) generalized Lotka-Volterra systems

$$\tau^{(\nu)} \frac{\mathrm{d}\alpha^{(\nu)}_{k_\nu}(t)}{\mathrm{d}t} = \alpha^{(\nu)}_{k_\nu}(t) \left(\sigma^{(\nu)}_{k_\nu} - \sum_{j_\nu=1}^{n_\nu} \rho^{(\nu)}_{k_\nu j_\nu}(t) \alpha^{(\nu)}_{k_\nu}(t) \right) . \tag{1}$$

The population amplitudes $\alpha^{(\nu)}_{k_\nu}(t)$ of level ν prescribe the control parameters of the higher level $\nu - 1$ by virtue of

$$\rho^{(\nu)}_{k_\nu j_\nu}(t) = \sum_{l_{\nu+1}=1}^{m_{\nu+1}} \alpha^{(\nu+1)}_{l_{\nu+1}}(t) r^{(\nu)}_{k_\nu j_\nu l_{\nu+1}} \tag{2}$$

where the constants $r^{(\nu)}_{k_\nu j_\nu l_{\nu+1}}$ serve as parameter templates.

Finally, the amplitudes $\alpha^{(\nu)}_{k_\nu}(t)$ recruit a hierarchy of modes $V^{(\nu)}_{k_\nu}(x)$ in the neural field such that

$$V(s) = V(x,t) = \sum_{\nu=1}^{\infty} \sum_{k_\nu=1}^{n_\nu} \alpha^{(\nu)}_{k_\nu}(t) V^{(\nu)}_{k_\nu}(x) . \tag{3}$$

Assuming a system of bi-orthogonal modes $V^{(\nu)^+}_{k_\nu}(x)$ with

$$\int_\Omega V^{(\nu)^+}_{k_\nu}(x)\, V^{(\nu)}_{j_\nu}(x)\, \mathrm{d}x = \delta_{k_\nu j_\nu} \tag{4}$$

we obtain from (3)

$$\alpha^{(\nu)}_{k_\nu}(t) = \int_\Omega V^{(\nu)^+}_{k_\nu}(x)\, V(x,t)\, \mathrm{d}x . \tag{5}$$

Next we convert the Lotka-Volterra differential equations into their corresponding integral equations by formally integrating

$$\alpha^{(\nu)}_{k_\nu}(t) = \frac{1}{\tau^{(\nu)}} \int_{-\infty}^{t} \alpha^{(\nu)}_{k_\nu}(t_\nu) \left(\sigma^{(\nu)}_{k_\nu} - \sum_{j_\nu=1}^{n_\nu} \rho^{(\nu)}_{k_\nu j_\nu}(t_\nu) \alpha^{(\nu)}_{j_\nu}(t_\nu) \right) \mathrm{d}t_\nu \tag{6}$$

and obtain a recursion equation

$$\alpha^{(\nu)}_{k_\nu}(t) = \frac{1}{\tau^{(\nu)}} \int_{-\infty}^{t} \alpha^{(\nu)}_{k_\nu}(t_\nu) \left(\sigma^{(\nu)}_{k_\nu} - \sum_{j_\nu=1}^{n_\nu} \sum_{l_{\nu+1}=1}^{m_{\nu+1}} r^{(\nu)}_{k_\nu j_\nu l_{\nu+1}} \alpha^{(\nu+1)}_{l_{\nu+1}}(t_\nu) \alpha^{(\nu)}_{j_\nu}(t_\nu) \right) \mathrm{d}t_\nu . \tag{7}$$

Now we consider two levels in the hierarchy (1). For the first level we have the recursion equation (7)

$$
\begin{aligned}
\alpha_{k_1}^{(1)}(t) &= \frac{1}{\tau^{(1)}}\int_{-\infty}^{t}\alpha_{k_1}^{(1)}(t_1)\left(\sigma_{k_1}^{(1)}-\sum_{j_1=1}^{n_1}\sum_{l_2=1}^{m_2} r_{k_1 j_1 l_2}^{(1)}\alpha_{l_2}^{(2)}(t_1)\alpha_{j_1}^{(1)}(t_1)\right)\mathrm{d}t_1\\
&= \frac{\sigma_{k_1}^{(1)}}{\tau^{(1)}}\int_{-\infty}^{t}\alpha_{k_1}^{(1)}(t_1)\,\mathrm{d}t_1-\frac{1}{\tau^{(1)}}\sum_{j_1=1}^{n_1}\sum_{l_2=1}^{m_2} r_{k_1 j_1 l_2}^{(1)}\int_{-\infty}^{t}\alpha_{k_1}^{(1)}(t_1)\alpha_{j_1}^{(1)}(t_1)\times\\
&\quad\left(\frac{\sigma_{l_2}^{(2)}}{\tau^{(2)}}\int_{-\infty}^{t}\alpha_{l_2}^{(2)}(t_2)\,\mathrm{d}t_2-\frac{1}{\tau^{(2)}}\sum_{j_2=1}^{n_2}\rho_{l_2 j_2}^{(2)}\int_{-\infty}^{t}\alpha_{l_2}^{(2)}(t_2)\alpha_{j_2}^{(2)}(t_2)\,\mathrm{d}t_2\right)\mathrm{d}t_1\\
&= \frac{\sigma_{k_1}^{(1)}}{\tau^{(1)}}\int_{-\infty}^{t}\alpha_{k_1}^{(1)}(t_1)\,\mathrm{d}t_1-\frac{\sigma_{l_2}^{(2)}}{\tau^{(1)}\tau^{(2)}}\sum_{j_1=1}^{n_1}\sum_{l_2=1}^{m_2} r_{k_1 j_1 l_2}^{(1)}\int_{-\infty}^{t}\int_{-\infty}^{t}\alpha_{k_1}^{(1)}(t_1)\alpha_{j_1}^{(1)}(t_1)\alpha_{l_2}^{(2)}(t_2)\,\mathrm{d}t_1\mathrm{d}t_2+\\
&+ \frac{1}{\tau^{(1)}\tau^{(2)}}\sum_{j_1=1}^{n_1}\sum_{l_2=1}^{m_2}\sum_{j_2=1}^{n_2} r_{k_1 j_1 l_2}^{(1)}\rho_{l_2 j_2}^{(2)}\int_{-\infty}^{t}\int_{-\infty}^{t}\alpha_{k_1}^{(1)}(t_1)\alpha_{j_1}^{(1)}(t_1)\alpha_{l_2}^{(2)}(t_2)\alpha_{j_2}^{(2)}(t_2)\,\mathrm{d}t_1\mathrm{d}t_2
\end{aligned}
$$

with

$$
\alpha_{k_2}^{(2)}(t)=\frac{\sigma_{k_2}^{(2)}}{\tau^{(2)}}\int_{-\infty}^{t}\alpha_{k_2}^{(2)}(t_2)\,\mathrm{d}t_2-\frac{1}{\tau^{(2)}}\sum_{j_2=1}^{n_2}\rho_{k_2 j_2}^{(2)}\int_{-\infty}^{t}\alpha_{k_2}^{(2)}(t_2)\alpha_{j_2}^{(2)}(t_2)\,\mathrm{d}t_2\,.
$$

Next we insert these amplitudes into (3) to obtain

$$
\begin{aligned}
V(x,t) &= \sum_{k_1=1}^{n_1}\alpha_{k_1}^{(1)}(t)V_{k_1}^{(1)}(x)+\sum_{k_2=1}^{n_2}\alpha_{k_2}^{(2)}(t)V_{k_2}^{(2)}(x)\\
&= \sum_{k_1=1}^{n_1}\left(\frac{\sigma_{k_1}^{(1)}}{\tau^{(1)}}\int_{-\infty}^{t}\alpha_{k_1}^{(1)}(t_1)\,\mathrm{d}t_1-\right.\\
&- \frac{\sigma_{l_2}^{(2)}}{\tau^{(1)}\tau^{(2)}}\sum_{j_1=1}^{n_1}\sum_{l_2=1}^{m_2} r_{k_1 j_1 l_2}^{(1)}\int_{-\infty}^{t}\int_{-\infty}^{t}\alpha_{k_1}^{(1)}(t_1)\alpha_{j_1}^{(1)}(t_1)\alpha_{l_2}^{(2)}(t_2)\,\mathrm{d}t_1\mathrm{d}t_2+\\
&+ \left.\frac{1}{\tau^{(1)}\tau^{(2)}}\sum_{j_1=1}^{n_1}\sum_{l_2=1}^{m_2}\sum_{j_2=1}^{n_2} r_{k_1 j_1 l_2}^{(1)}\rho_{l_2 j_2}^{(2)}\int_{-\infty}^{t}\int_{-\infty}^{t}\alpha_{k_1}^{(1)}(t_1)\alpha_{j_1}^{(1)}(t_1)\alpha_{l_2}^{(2)}(t_2)\alpha_{j_2}^{(2)}(t_2)\,\mathrm{d}t_1\mathrm{d}t_2\right)V_{k_1}^{(1)}(x)+\\
&+ \sum_{k_2=1}^{n_2}\left(\frac{\sigma_{k_2}^{(2)}}{\tau^{(2)}}\int_{-\infty}^{t}\alpha_{k_2}^{(2)}(t_2)\,\mathrm{d}t_2-\frac{1}{\tau^{(2)}}\sum_{j_2=1}^{n_2}\rho_{k_2 j_2}^{(2)}\int_{-\infty}^{t}\alpha_{k_2}^{(2)}(t_2)\alpha_{j_2}^{(2)}(t_2)\,\mathrm{d}t_2\right)V_{k_2}^{(2)}(x)
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \int_{-\infty}^{t} \sum_{k_1=1}^{n_1} \frac{\sigma_{k_1}^{(1)}}{\tau^{(1)}} \alpha_{k_1}^{(1)}(t_1) V_{k_1}^{(1)}(x)\,\mathrm{d}t_1 - \\
- \quad & \int_{-\infty}^{t}\int_{-\infty}^{t} \sum_{k_1=1}^{n_1}\sum_{j_1=1}^{n_1}\sum_{l_2=1}^{m_2} \frac{\sigma_{l_2}^{(2)} r_{k_1 j_1 l_2}^{(1)}}{\tau^{(1)}\tau^{(2)}} \alpha_{k_1}^{(1)}(t_1)\alpha_{j_1}^{(1)}(t_1)\alpha_{l_2}^{(2)}(t_2) V_{k_1}^{(1)}(x)\,\mathrm{d}t_1\mathrm{d}t_2 + \\
+ \quad & \int_{-\infty}^{t}\int_{-\infty}^{t} \sum_{k_1=1}^{n_1}\sum_{j_1=1}^{n_1}\sum_{l_2=1}^{m_2}\sum_{j_2=1}^{n_2} \frac{r_{k_1 j_1 l_2}^{(1)} \rho_{l_2 j_2}^{(2)}}{\tau^{(1)}\tau^{(2)}} \alpha_{k_1}^{(1)}(t_1)\alpha_{j_1}^{(1)}(t_1)\alpha_{l_2}^{(2)}(t_2)\alpha_{j_2}^{(2)}(t_2) V_{k_1}^{(1)}(x)\,\mathrm{d}t_1\mathrm{d}t_2 + \\
+ \quad & \int_{-\infty}^{t} \sum_{k_2=1}^{n_2} \frac{\sigma_{k_2}^{(2)}}{\tau^{(2)}} \alpha_{k_2}^{(2)}(t_2) V_{k_2}^{(2)}(x)\,\mathrm{d}t_2 - \int_{-\infty}^{t} \sum_{k_2=1}^{n_2}\sum_{j_2=1}^{n_2} \frac{\rho_{k_2 j_2}^{(2)}}{\tau^{(2)}} \alpha_{k_2}^{(2)}(t_2)\alpha_{j_2}^{(2)}(t_2) V_{k_2}^{(2)}(x)\,\mathrm{d}t_2 \,.
\end{aligned}
$$

Then we eliminate the amplitudes by means of relation (5), yielding the series

$$
\begin{aligned}
V(x,t) \quad = \quad & \int_\Omega \mathrm{d}x_1 \int_{-\infty}^{t} \mathrm{d}t_1 K_1(x,x_1) V(x_1,t_1) \\
- \quad & \int_\Omega \mathrm{d}x_1 \int_\Omega \mathrm{d}x_2 \int_{-\infty}^{t} \mathrm{d}t_2 K_2(x,x_1,x_2) V(x_1,t_1) V(x_2,t_1) \\
- \quad & \int_\Omega \mathrm{d}x_1 \int_\Omega \mathrm{d}x_2 \int_\Omega \mathrm{d}x_3 \int_{-\infty}^{t} \mathrm{d}t_1 \int_{-\infty}^{t} \mathrm{d}t_2 \\
& \times K_3(x,x_1,x_2,x_3) V(x_1,t_1)\, V(x_2,t_1)\, V(x_3,t_2) \\
+ \quad & \int_\Omega \mathrm{d}x_1 \int_\Omega \mathrm{d}x_2 \int_\Omega \mathrm{d}x_3 \int_\Omega \mathrm{d}x_4 \int_{-\infty}^{t} \mathrm{d}t_1 \int_{-\infty}^{t} \mathrm{d}t_2 \\
& \times K_4(x,x_1,x_2,x_3,x_4)\, V(x_1,t_1)\, V(x_2,t_1)\, V(x_3,t_2)\, V(x_4,t_2)
\end{aligned}
\tag{8}
$$

with the kernels

$$
\begin{aligned}
K_1(x,x_1) \quad = \quad & \sum_{k_1=1}^{n_1} \frac{\sigma_{k_1}^{(1)}}{\tau^{(1)}} V_{k_1}^{(1)^+}(x_1) V_{k_1}^{(1)}(x) + \sum_{k_2=1}^{n_2} \frac{\sigma_{k_2}^{(2)}}{\tau^{(2)}} V_{k_2}^{(2)^+}(x_1)\, V_{k_2}^{(2)}(x) \\
K_2(x,x_1,x_2) \quad = \quad & \sum_{k_2=1}^{n_2}\sum_{j_2=1}^{n_2} \frac{\rho_{k_2 j_2}^{(2)}}{\tau^{(2)}} V_{k_2}^{(2)^+}(x_1) V_{j_2}^{(2)^+}(x_2) V_{k_2}^{(2)}(x) \\
K_3(x,x_1,x_2,x_3) \quad = \quad & \sum_{k_1=1}^{n_1}\sum_{j_1=1}^{n_1}\sum_{l_2=1}^{m_2} \frac{\sigma_{l_2}^{(2)} r_{k_1 j_1 l_2}^{(1)}}{\tau^{(1)}\tau^{(2)}} V_{k_1}^{(1)^+}(x_1) V_{j_1}^{(1)^+}(x_2) V_{l_2}^{(2)^+}(x_3) V_{k_1}^{(1)}(x) \\
K_4(x,x_1,x_2,x_3,x_4) \quad = \quad & \sum_{k_1=1}^{n_1}\sum_{j_1=1}^{n_1}\sum_{l_2=1}^{m_2}\sum_{j_2=1}^{n_2} \frac{r_{k_1 j_1 l_2}^{(1)} \rho_{l_2 j_2}^{(2)}}{\tau^{(1)}\tau^{(2)}} V_{k_1}^{(1)^+}(x_1) V_{j_1}^{(1)^+}(x_2) V_{l_2}^{(2)^+}(x_3) \times \\
& V_{j_2}^{(2)^+}(x_4) V_{k_1}^{(1)}(x)
\end{aligned}
$$

Interestingly the generalized Volterra series kernels are time-independent. Since neural field equations can be written in the same form as Eq. (8), this result shows that hierarchies of Lotka-Volterra systems are included in the neural field description. Again we point out that the resulting kernels are linear combinations of dyadic products as introduced in the main manuscript (see also R. Veltz and O. Faugeras, Local/global analysis of the stationary solutions of some neural field equations. SIAM Journal on Applied Dynamical Systems 9:954 – 998 (2010)) .