Examples of inquiry-based activities with reference to different theoretical frameworks in mathematics education research

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This collection of inquiry-based activities in mathematics teaching is an appendix to the paper "Conceptualizing inquiry-based education in mathematics" by the same authors published in *ZDM* (Artigue and Blomhøj, 2013). In the following we refer to this paper simply by the paper. The examples are selected and discussed to illustrate how the different theoretical frameworks analysed in the paper offer different perspectives for the conceptualization and implementation of inquiry-based mathematics education (IBME).

1. Taxicab geometry – The problem-solving tradition

There is a long and strong tradition in mathematics education concerning problem solving and its role in mathematics teaching (see Schoenfeld 1992 for a survey of this research tradition). Here we present an example of how students' problem-solving activities within a system of interrelated problems presented in a certain context can constitute a landscape of investigation.

The notion of a landscape of investigation was developed by Ole Skovsmose and analysed in detail in Alrø and Skovsmose (2002, pp. 46–67). The main idea is to capture the minimal changes in the learning environment, which can cause the students to notice and engage in inquiring the relations between the problems and the context, instead of just dealing with the tasks as isolated problems. By means of presenting to the students a context, which gives meaning to the system of tasks, the students are invited into an area of mathematical investigations in which they can organize their findings and formulate new problems. The possibility for the students to relate to the context based on their experiences and to talk with each other about the context, the problems, their ideas and their findings in their own words is pinpointed as crucial in order for a system of tasks to form a landscape of investigation. Alrø and Skovsmose (p. 50) distinguish between landscapes of investigation with reference to pure mathematics, semi-reality and actual reality. Here we present an example of a landscape of investigation with reference to a semi-reality, namely a system of tasks related to the so-called "taxicab geometry". Depending on educational level (grade 5 to university level), the context can be presented to the students as doing mathematics in a strange city, where the streets form a perpendicular lattice, where people live only in the intersections of streets, and where the distance between points of intersection is measured by counting the number of blocks you have to pass to get from one point to the other (see Fig. 1).

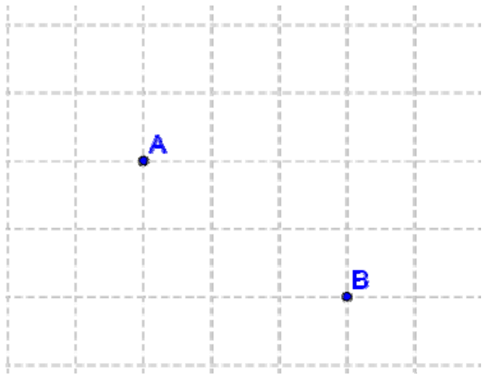

Fig. 1 Two points in the city with distance 5. $T(A, B) = 5$

Depending on grade level, a symbolic representation of the taxicab-distance between two points *A* and *B* can be introduced as in Fig. 1: $T(A, B) = 5$.

After having set the scene for the students' work in this manner, a first system of tasks can be presented to the students with reference to Fig. 1:

(A1) If possible, draw round tours starting and ending at *A* of length 4, 8, 9 and 12.
(A2) If possible, find points which are the same distance from both *A* and *B*.
(A3) What if you change the position of *B* in A2?

The dialogue between the teacher and the students during their work with the tasks is important for supporting the students to enter into the landscape of investigation. In a grade 8 class a dialogue could run as follows:

T: Have you made the round tours?
P1: Yes, but if we are not allowed to make a U-turn between two points we cannot make one with length 9.
T: Yeah, you are right. It is not allowed to turn between points.
P2: Then it is not possible.
T: Are you sure?
P1: We think we are sure – is it not right?
T: Okay, why do you think it is impossible with 9?
P2: Maybe because 9 is odd – the others are even.
T: Good suggestion. Maybe you could try some other odd numbers.
[After some minutes the teacher returns to the pupils.]
T: Have you found any round tours with odd length?
P1: No, it is not possible.
T: Can you formulate a rule?
P2: It is impossible to have an odd round tour.
T: Good, so what can be said about a round tour?
P1: It will have even length.
T: Very good – nice to know, but can you prove it?
[After some minutes the pupils ask the teacher for help.]
T: Each time you go one block north you have to go one block south somewhere on the tour, right.
P2: Yes, and the same with east and west.

T: Exactly, so if you go x blocks north and y blocks east on the tour how can you express the length?

After a while the pupils have written the expression $2x + 2y$ for the length of a round tour. They argue that a sum of two even numbers is even, and that this proves the rule.

The following system of tasks referring to the same context can be introduced later:

(B1) Mark all the points which are distance 3 from *A*.
(B2) How many such points are there? Find a suitable name for such a pattern.
(B3) What happens if you change the distance to 4? Or 5?
(B4) Build a formula for the number of points which are distance *r* from *A*.
(B5) Build a formula for the number of points which are distance less than *r* from *A*.

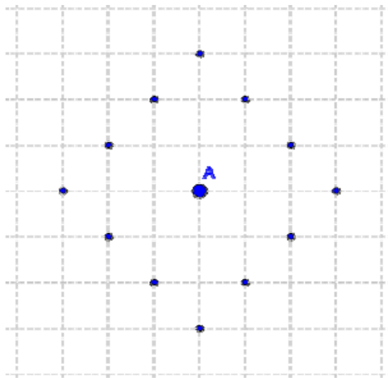

Fig. 2 The 12 points which are distance 3 from the point *A* for a taxicab circle with radius 3

Again depending on class level, the following symbolic representations may be introduced with reference to Fig. 2, where *r* belongs to $\boldsymbol{N}_0$:

$N(r)$: Number of points distance *r* from a given point – points on the taxicab circle.
$P(r)$: Number of points distance less than *r* from a given point – points in the circle.

Using their language description, such as "The number of points within a circle must be the sum of the points within the circle one less and the points on that circle", from grade 6 the students can develop a table such as depicted in Fig. 3.

r	0	1	2	3	4
$N(r)$	1	4	8	12	16
$P(r)$	0	1	5	13	25

Fig. 3 Table with number of points on $N(r)$ and inside $P(r)$ for a taxicab circle with radius *r*

Many students will investigate the pattern of such a table by themselves, and – with support from the teacher – be able to formulate the recursive structure of $P(r)$. From grade 8 the students can work with expressing the notion of a function and symbolical representations such as:

$N(r) = 4r$ for $r > 0$ and $N(0)=1$; $P(r) = P(r-1) + N(r-1)$ for $r > 0$ and with $P(0) = 0$

Given time and support, students in grade 8 may be able to find an expression equal to $P(r) = 2r^2 - 2r + 1$ for the number of points inside a taxicab circle with radius r, either from geometrical reasoning or from working with the numbers in the table. At upper secondary level the formula can be deduced from the recursive expression.

In the continuation of inquiry in this landscape of investigation the students can easily – if encouraged by the teacher – pose new interesting questions themselves based on their findings. For example, given a city of a certain size and shape, where should three schools (of equal size) be placed in order to cover the area best?

At university level the students could be challenged to reflect on the properties which define a metric, and experience taxicab geometry as just one example of a metric space in which the distances belong to $\boldsymbol{N}_0$.

In a problem-solving approach to IBME, the mathematical richness of the problems addressed, their relations to mathematical ideas and concepts, and the possibility for the students to engage in the problem-solving activities in their own ways and to obtain results which can lead to new problems and organizations of findings, become important theoretical issues as well in the practice of teaching.

2. The rope triangle – Theory of Didactical Situations (TDS)

TDS literature provides many exemplars of situations illustrating the form that inquiry can take in such an approach, from the most elementary grades up to university, and some of these such as the "Puzzle situation" or the "Race to 20" are well-known. We choose not to use such paradigmatic examples here but rather a situation designed for grade 5–6 students in a Danish school in the framework of the PRIMAS European project. This situation was designed with the aim that the students should construct the triangle inequality as a general property for triangles. In groups of four the students are presented with a rope with 12 equidistant knots forming a ring of length 12 units. Each group has its own rope. The task is introduced to the students in the following manner: "You should make triangles with the rope in the schoolyard, but only triangles which have a knot at each of their three vertices are allowed. You can make the triangles by three of you each holding a knot. For each triangle you find you should make a sketch with the lengths of the three sides on the sketch. Your task is to find all possible triangles that can be made in this way."

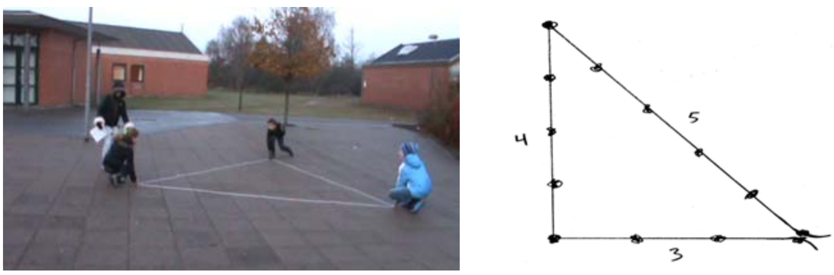

Fig. 3 The situation used in the Danish programme under the PRIMAS project in a grade 6 class. The figure shows a still from a video where a group of students has constructed an equilateral triangle with side length 4. The group also found the 3-4-5 triangle, as sketched

As did the group depicted in Fig. 3, groups normally quite easily find two of the three possible triangles: 4-4-4 and 3-4-5. For some reason the 2-5-5 triangle seems to be more difficult to find. However, many groups search for other triangles, and some groups insistently try to construct a 2-4-6 or a 3-3-6 triangle. This phase (30–40 minutes) functions as an a-didactic phase of action where students interact with the milieu in order to find triangles fulfilling the requirements. The characteristics of the milieu and the limited number of solutions make it likely that the class will find all three possible triangles.

Thereafter, the class is gathered in the classroom for formulating their findings. All groups can be expected to contribute with possible triangles. Eventually all three possible triangles are drawn on the blackboard. It might be that the rope is used again to illustrate one of the triangles. In this phase of formulation, some specific semiotic code (for instance 3-4-5, 4-4-4 and 2-5-5) is introduced, with the help of the teacher, as shorthand for representing the triangles. The teacher then moves to a phase of validation by asking the class if there could be other possible triangles. Such a question indeed leads to envisaging and finding ways to reject other possibilities – for instance a 2-4-6 (or 3-3-6) triangle.

The rope is then reintroduced, but now in an "open" version with knots at both ends and length 12, which makes a change in the material milieu. This time the class may watch two students holding the rope with six units between them and leaving respectively four and two units at each end of the rope. In order for the two ends to be joined together by a third and a fourth student they have to meet on the side of length 6. Based on such experience the class, in dialogue with the teacher, reaches the conclusion that a 2-4-6 triangle is impossible. The validation is here a pragmatic validation through action, and each new case envisaged needs a specific validation. At this stage, the teacher may ask the class if they can formulate a rule which the three lengths have to fulfil in order for a triangle with these lengths to be possible. This opens the way towards systematic reasoning and generalization with the help of the teacher, as illustrated in the following dialogue:

S1: The sum of the lengths should be 12.
T: Right, but that is not enough since 2 + 4 + 6 = 12.
S2: The longest side must not be longer than 5.
T: Good suggestion! Why not?
S2: Because 6 is not possible and 7 makes it even more difficult for the ends to meet – impossible in fact.
T: [The teacher makes a drawing of an "open triangle" with sides 2-3-7.] Do you all agree?
T: So you are right, in all three triangles the longest sides are 5 or 4 – well 4-4-4 is equilateral so there is no side longer than 4.
T: Are we sure that we have found all triangles with no side longer than 5?
T: 5 for one side – how much does that leave for the two other sides?
S3: 7.
T: Yes, and how can seven be written as a sum of two natural numbers less than 6?
S4: 2 + 5 and 3 + 4. And these two we have already.
T: And we have also 4-4-4. Could there be others with 4 as the maximum?
S2: No, one of the three numbers has to be higher if the sum should be 12.
T: Very good – we got them all – there are these three!

[The teacher points to the blackboard.]

T: What if the rope was longer – with more knots? Can we formulate a general rule?

T: [The teacher draws a "general" triangle *ABC* at the blackboard with side lengths *a*-*b*-*c*.] Let us say that *c* is the longest side. What can we say about *a* and *b* in relation to *c*?

S2: Together *a* and *b* should be longer than *c*.

T: Right. $a + b > c$ or $AC + CB > AB$. This is a general rule for all triangles!

This example illustrates the different phases of didactical games mentioned above: devolution, a-didactic action, formulation, validation and institutionalization. The students are given mathematical responsibility in their investigation, and in the formulation and validation of their findings. However, the proof that all possible triangles have been found is taking place during the teacher's dialogue with the class. It is a didactic situation to which the students contribute, building on their experiences from their interaction with the milieu. The milieu moves from one phase to the next and is progressively enriched by the results already obtained, which helps the students engage in formulation and validation.

The example illustrates how inquiry activities in a TDS approach are carefully designed to be a didactical vehicle for a particular mathematical learning objective. Inquiry activities play an important role in TDS in establishing a-didactical situations in which students can develop experiences which can serve as motivation and cognitive basis for their mathematical learning. This is very much in line with Dewey's original ideas. However, we also see that the clear orientation of the inquiry process through the design and management of the situation can make other ideas important in Dewey's philosophy difficult to incorporate: engaging students in open inquiry activities, giving them important responsibility in the choice of the questions addressed and the development of the inquiry process, and cultivating their autonomy as learners.

The activity was designed to produce what could be called a fundamental situation for the construction of the triangle inequality as a general property for all triangles. This means that the triangle inequality should be the only rational way for explaining the findings from the inquiry activities. However, it turns out that one of the students suggests a more particular rule (no side longer than 5), which explains the findings. In order to overcome this limitation of the situation, the teacher asks the class directly for a generalization which could establish a basis for the institutionalizing of the triangle inequality, and apparently it works. For us this illustrates an important didactical challenge related to the implementation of TDS in the practice of mathematics teaching – fundamental situations are very difficult to accomplish in practice. Bringing the inquiry elements of TDS into focus very much accentuates this challenge and, as a consequence, highlights the importance to be attached to the quality of the dialogue and joint action between students and teacher (Sensevy 2011).

3. From the school theatre to the percentage bar – Realistic Mathematics Education (RME)

When teaching the concept of percentage in grade 5 classes in a RME approach this activity could play a central role in introducing the student to the percentage bar as a standard model for them to work with in their future mathematizations (Van den Heuvel-Panhuizen 2003).

The students are asked in pairs or small groups to determine how large a percentage of the seats in the school theatre, which is well known to the students, they would expect to be occupied at three different types of performance which are imagined to take place in the near future. The students are asked: "How busy will the theatre be during each performance? Colour the part of the theatre that you think will be occupied and write down the percentage of the seats used in each of the three cases." The students are given a sheet of paper as depicted in Fig. 4.

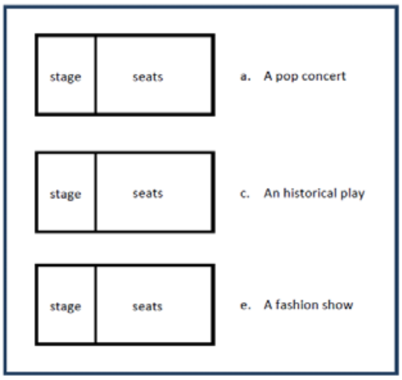

Fig. 4 The sheet given to the students

This is an example of an explorative activity to support students in building models (i.e. the bar model) based upon their prior ideas and experiences. The activity will result in many different solutions because the students are judging the popularity of the different events differently and because some solutions will take special aspects into account, for example some groups argue that at the pop concert there should be a space left free right in front of the stage for dancing, while other groups consider which seats would be best for the different events, instead of just filling up the theatre backwards from the stage. Some students draw rows of seats with the same number of seats in each row, typically ten or twenty, which allow them to represent very precisely a certain percentage of occupied seats – say 55% represented by five and a half rows out of ten rows. For sure, some students also come up with the extreme solutions of 100% (for the pop concert) with the entire seat area painted black and 0% (for the historical play) with no colour on the figure at all.

Building on their experiences from such exploring activities and discussions of solutions and their representations with the class, the students can work with a system of tasks, including open inquiry activities as well as more closed practising activities, which guide them to reinvent the mathematics of percentages. The bar model initially emerges as a model of specific situations and turns into a model for thinking and reasoning about percentages and connecting them with fractions. The aim for the students' learning is a fully developed bar model for representing percentages in problem situations (Van den Heuvel-Panhuizen 2003, p. 22) as depicted in Fig. 5.

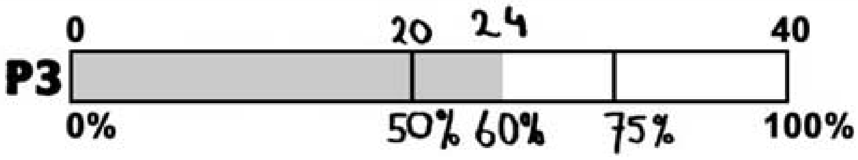

Fig. 5 The fully developed bar model, as the "occupation meter" for a parking area with 40 parking places

As Van den Heuvel-Panhuizen (2003, p. 9) points out:

> The power of this model is that it develops alongside both the teaching and the students: from a drawing that represents a context related to percentage to a strip for estimation and reasoning to an abstract tool that supports the use of percentage as an operator.

This example illustrates that in RME the students' inquiry activities play crucial roles both in the process of forming the students' experiences with using mathematical concepts and representations in meaningful real-life problem situations through horizontal mathematization, and in the process of reinventing the mathematical meaning of these concepts and representations through vertical mathematization. In RME, inquiry activities are guided by the long-term perspective of supporting the students' reconstruction of certain learning trajectories concerning important mathematical concepts and models of representation.

4. The asthma drug project – The mathematical modelling perspective

This project was originally designed and carried out by two teachers teaching mathematics and chemistry at the same Danish gymnasium – a programme of three years of upper secondary schooling (grades 10–12) – as part of an in-service course on project work and mathematical modelling (Blomhøj and Kjeldsen 2006, pp. 168–172). The project takes its point of departure as the optimization of medication and was inspired by an authentic case of modelling for controlling the concentration of drugs during anaesthesia presented at the course. However, in this project the case was asthma medication and the target group was first year students – grade 10. The project is designed for students who have not yet learned about the exponential function but have some previous experience with constant relative growth and decline. In fact, the two main rationales for the project were to develop the students' modelling competence and at the same time to motivate and prepare them for learning about the exponential function. With variations, this project has been carried out by different teachers in several classes since the in-service course. In general the project seems to challenge most students at this grade at a very appropriate level concerning both their mathematical understanding and their modelling competence. Typically the project runs over 8–10 lessons (45 minutes each) including introduction and students' presentations.

In the project the scene is set for the students' work by giving them the task of helping a medical doctor to develop an optimal medication schema for an asthma patient. In groups of three or four the students should write a letter to the doctor explaining in ordinary language their proposed medication schema and its rationale. Together with the letter, all data analyses and calculations should be documented and explained in detail in a report possibly to be used at an oral examination.

The students are given data showing the decline of the concentration of the asthma drug theophylline in the bloodstream for one particular patient. The data show the concentration as a function of the time in hours after an injection (see Fig. 6). The data are not authentic but designed deliberately to ease the way for the students in their work to describe the data mathematically. The work of the groups is regulated by means of the following more specific questions to be addressed in the reports: “(1) Describe the decline of the concentration of theophylline in the data mathematically and with normal language. (2) Suggest a medication schema with constant dose D and a fixed time interval T between injections so that the concentration stays in the interval 5–15 mg/l after three injections. (3) What is your suggestion as to whether the first dose, D_0, could be different from D? (4) What criteria should a good medication schema fulfil? (5) What additional aspects should be taken into account for the medication schema to be applicable to our patients?”

Time (hours)	0	2	4	6	8	10	12	14	16	18
Conc. (mg/l)	10	7	5	3.5	2.5	1.9	1.3	0.9	0.6	0.5

Fig. 6 The decline in the concentration of theophylline in the bloodstream in a patient in mg/l as a function of hours after an injection resulting in an initial concentration of 10 mg/l

Most of the groups manage to set up a function describing the data by observing that $C(t + 4) = ½C(t)$ and considering that $C(t + 1)= kC(t)$, where k^4 should be ½, but only very few express their observations symbolically. However, most groups end up describing the decline in the concentration by the function $C(t) = 10(0.84)^t$. With different degrees of support from the teacher the groups are able to set up a model, for example in a spreadsheet, which allows them to experiment with changing the dose and the time interval to reach a suitable medication schema (see Fig. 7).

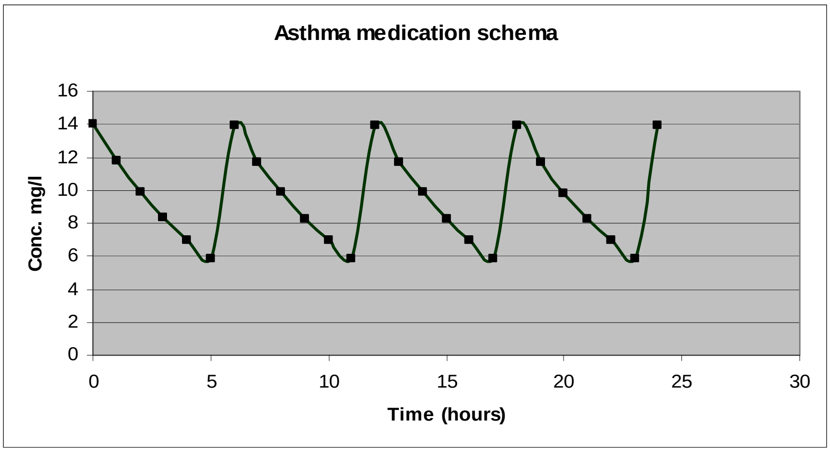

Fig. 7 The drug concentration during 24 hours after an injection, which results in an initial concentration of 14 mg/l and followed by injections each 6 hours with a dose giving an additional 9 mg/l.

In their work with designing the medication schema many groups realize that it is important for the patients that the injections take place at the same time of the day each day during the treatment, and that the time interval should be as long as possible to minimize the number of injections. They also recognize that it is difficult to keep the concentration in the therapeutic interval with only three injections per day. The medication schema shown in Fig. 7 is with four injections – one each six hours – and with a dose equal to 9 mg/l.

Using a spread sheet, it is easy for the groups to experiment with their model, and during this work a new question arises. Would it be better for the patient if there was a longer time interval during the night and what should be the dose before the night interval? Some groups experiment with their spread sheet model to answer the question, while other groups make a calculation using their function for the decline of concentration between injections. They ask questions such as: "How much drug – measured in concentration – should be given in order to compensate for the decline over one interval?" Also, some groups express critical reflections concerning the situation given such as: "Why is the drug given by injections and not by tablets?" and "Would it not be better with tablets since the drug is released gradually?"

In relation to question five many reports reveal interesting reflections such as: "Before the model is used for other patients the same type of data that we had should be measured for each patient. There might be individual differences according to sex, body mass and physical activity and other habits" and "It should be possible to make individual medication schemas for other patients, but you need data for each person."

We find this project to be exemplary for illustrating how students can develop a simple mathematical model which allows them to investigate a relevant real-life problem by experimenting with their model. The project really involves the students in inquiry activities and in modelling. Most groups work with all the sub-processes (b)–(f) of the modelling cycle (see Fig. 3 in the paper). The formulation of the problem is more or less given by the data and the questions to be addressed in the students' reports. However, in the first project session, the teachers discuss with their classes how to formulate a problem based on the general description of the real problem situation. Data and sub-questions are given to the students only after the class has formulated a common problem to investigate. Also, with respect to the duality between seeing modelling as a vehicle for the learning of mathematics and modelling competence as an educational end for mathematics teaching, we find this project to be exemplary.

5. Teaching mathematics from magnitudes – Anthropological Theory of Didactics

Between 2010 and 2012 members of the AMPERE group in the IREM of Poitiers developed and experimented with a series of PSR (Programmes of Study and Research) in the sense of ATD. These PSR organize a substantial part of teaching in grade 6 from inquiry projects around different types of magnitudes: angle, length, area, money, duration, volume. They all share a similar structure and we focus here on the area PSR (IREM, 2010). In coherence with the ATD vision of the inquiry process in the paper, emphasis is put on the three main questions which structure the programme and their raison d'être: How to compare areas? How to measure an area? How to calculate an area? These questions and their order indeed emerge from an epistemological analysis but they are then more precisely framed by considering the grade 6 curriculum.

The work carried out on each of these generating questions is expected to address different types of intra- and extra-mathematical problems and to support the development of different techniques and associated technologies. Comparing areas, for instance, promotes the development of cutting and re-assembling techniques, which enable students to solve a range of questions. Measuring or estimating areas through the use and refinement of grids allows them to solve new types of problems involving a wide diversity of forms, not necessarily polygonal. Calculating areas introduces the development and use of formulas as complementary and powerful tools. In the whole process, rectangles are given a crucial role and all these contribute to the conceptualization of area as a magnitude. Many different implementations of these ideas are of course possible and the publication of the IREM of Poitiers offers many resources and suggestions for inquiry which allow teachers to adapt the programme to their personal context and interests, including historical sources, human artefacts and media supports, together with data collected in experimentations of this PSR. Many of these suggestions incorporate, too, a critical dimension.

We cannot enter into a detailed presentation, but would like to briefly evoke how the PSR began in one of the reported implementations and how the work on the first generating question (How to compare areas?) developed. The PSR began by a classroom discussion around the following four questions: What is an area? When do we need areas? What do we need to know about areas? What are areas useful for? This discussion enabled students' visions and doubts to emerge, and questions not limited at all to the school context raised, as could be anticipated. It was an important source of information for the teacher, and it also helped her to introduce the theme to the students and to motivate them for it.

The work on the first generating question began, then, in the following way. Students were shown pictures of a table made of several connected pieces (Fig. 8), which could take different shapes according to the way these pieces were assembled. Then they were presented with a first problem to be solved in small groups after a moment of personal reflection:

"This square table is made of four equal parts. How to put these elements into one table, in order to have more space and accommodate the maximum number of guests?"

Fig. 8 The table problem

Different strategies emerged which were discussed at a second session and led to a collective definition of the work to be carried out for a further session (4 days after) including, for each proposed table, the realization and assemblage of pieces, a geometrical figure representing this assemblage with appropriate codes, and a descriptive text. At this session, the different productions were then presented by their authors, leading to a review of geometrical terms and basic properties of rectangular isosceles triangles, and were then compared. The comparison made clear that all tables did not offer the same possibilities for accommodating guests, that they had the same area but different perimeters. These were compared but it was also made clear that considering perimeters was not enough for solving the practical problem at stake.

From this first problem emerged new questions linked to the possibility of transforming a given geometrical shape into another, by cutting it into pieces and re-assembling the pieces without overlapping: a rectangle from a triangle or rhombus or a rectangle from a regular hexagon for instance. It was also made explicit that, in such a transformation, the area is an invariant, and that this invariance provides a technique for comparing areas. The technique was itself consolidated through variations around this same type of task: transforming polygons into rectangles having the same area. Note that different solutions generally exist for such tasks, which generates interesting collective discussions. For instance, interest in using axes of symmetry when they exist emerged from the comparison of students' productions and was given the status of a general method in the classroom (see Fig. 9).

Four sessions were devoted to this part of the PSR, in which links were explicitly made by the teacher between solutions found by the students for regular polygons and historical solutions, for instance a Chinese method for the dodecagon, which was especially rewarding for them. This phase culminated with the finding of a formula for the area of the regular polygons considered resulting from their transformation into a rectangle. Then students were given more freedom to create their own tasks. Figures including curves were eventually considered, showing the limitation of the cutting–assembling technique and leading to measuring and approximation entering the scene, in other words to addressing the second generating question: How to measure areas?

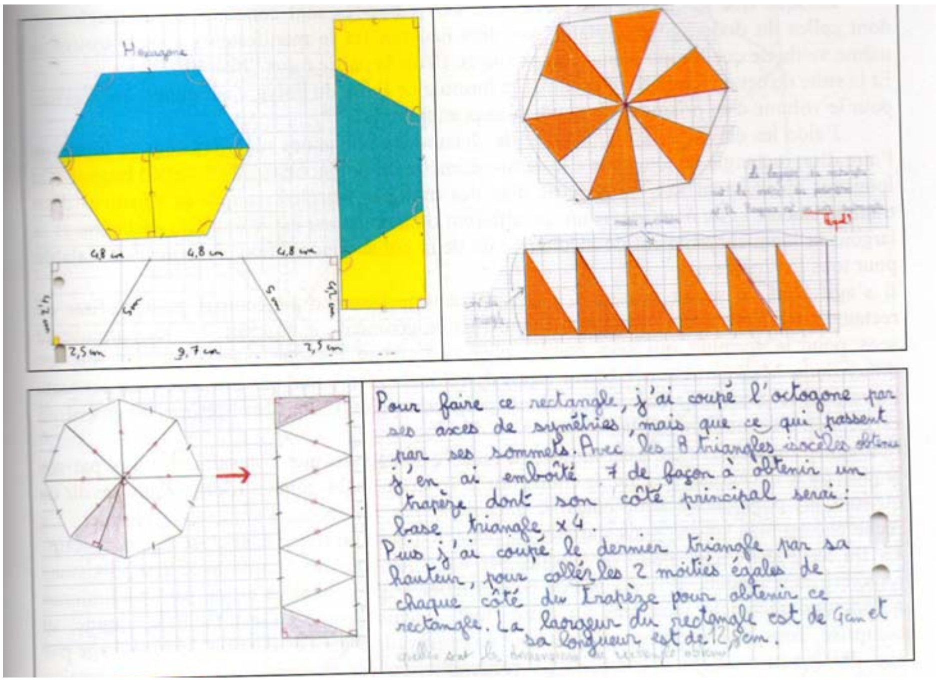

Fig. 9 Transformations of regular polygons into rectangles

Note that the cutting–assembling technique introduced was also used for approaching issues of re-parcelling, solving a practical problem of re-parcelling, and deciding if proposed exchanges of parcels were equitable or not.

This description provides just a limited view of the particular PSR, but we hope that it is sufficient for understanding how inquiry progressively develops in such a PSR: guided by a strong epistemological vision but nevertheless open; connecting mathematical and extra-mathematical concerns in its efforts to have students understand the *raisons d'être* of what they are taught and the use they can make of it;

paying attention to the curriculum but breaking with the ordinary structuration of it into separate chapters; not underestimating the importance of techniques which are an essential component of praxeologies. However, what is less visible in the description of this first part of the PSR is the functioning of the media–milieu dialectics, which did become more visible later.

6. Terrible small numbers and salmonella in eggs – Critical Mathematics Education

Inspired by problems with salmonella-infected eggs in Denmark in 1998–2000, a course of lessons on risk and responsibility was developed and tried out in grade 6–8 classes. The philosophy behind the course and its connection to the idea of education for democracy in a risk society as well as detailed analyses of some of the first courses can be found in Alrø and Skovsmose (2002, pp. 195–230). The project has been presented at many conferences and professional development courses and carried through with variations in many classes since then.

To set the scene for the course the class is presented with 500 photo tubes as "eggs" (see Fig. 10). Each "egg" contains a centicube. A blue one indicates that the egg is infected, while a yellow one indicates that the egg is free of salmonella. The students take samples and make estimates of the frequency of infected eggs in different loads of eggs and of the chance that dishes with different numbers of raw eggs will be infected. To begin with the students work with a statistical probability approach, but in some courses students were also introduced to simulation of the probability for infected dishes based on a given a priori probability for one randomly picked egg to be infected.

Fig. 10 A 10-package with at least one infected egg

In grades 8 and 9, classes calculate probabilities for complementary events (infected or non-infected dishes) assuming a binomial distribution of the number of infected eggs in a sample.

This course has given rise to many interesting dialogues among the students and between the teacher and the students. Here is an example where a group of students discuss how many samples they should take in order to be able to write a trustworthy declaration of the eggs:

S1: … I don't think we should take any samples.
S2: No … we could just save the money.
S3: Would you just sell the eggs?
S1: Yes, we cannot be sure anyway. So why not just sell a lot of cheap eggs?
S2: … and get very rich.
S3: Oh you are wicked … what if someone gets salmonella-infected!
S4 At least one can be a bit sure with samples …
S2 Right, then we should take two samples [of ten eggs each]. This will cost twenty kroner.
S1 No … the eggs also cost 0.50 kroner.
S3 Would you then just write "Tested" on the declaration?
S4 Then, I will not buy any of those eggs.
S2 Yes … then we just write: "Tested for salmonella".
S3 God knows if anybody is doing this.
S1 Don't you think that is illegal?

In the course the students are also challenged to calculate the probability of getting at least one infected egg in a cream with *n* eggs meant for *n* persons. A typical dialogue related to the students' judgements about how much more risky it is to serve cream for 100 persons at a wedding compared with serving cream for 6 persons at a dinner party could typically read as follows:

T: What you think about the chance of getting salmonella in a portion of cream with 6 eggs?
S1: 1% of our eggs are infected, so each time we pick an egg the chance is 0.01 – right?
T: Yes, how can you avoid salmonella in the cream?
S2: You need all six eggs to be good eggs.
S3: Then we have to multiply 0.99 six times – I think.
T: Right! That is the same as 0.99^6. What is then the chance that the cream is infected?
S3: It must be $1 - 0.99^6$. This gives 0.06 [by calculator].
T: What if the cream is with 100 eggs?
S1: Then it is just $1 - 0.99^{100} = 0.63$.
T: So how much more dangerous is the wedding cream compared with the cream for six persons?
S2: 10 times more dangerous.
T: The risk for the cream to be infected is 10 times as big, but how many people could be infected and become sick in each of the two cases?
S1: Could all 100 guests at wedding be infected if the cream is infected?
T: What do you think?

Based on such dialogues the students are challenged to make judgements about the risk of the different events, and at a certain point the teacher may introduce:

$$R(A) = C(A) \cdot P(A)$$

where *R* is the risk connected to some event *A*, which occurs with probability *P* and has the consequences *C* if it occurs. If *C* is measured by the number of persons who could be infected by salmonella, this simple model could be used to measure the risk of different events. Such experiences could eventually form a basis for discussing and critiquing this model for risk calculation. In cases where the probability is very small and the consequences are of catastrophic dimension, as is the case with nuclear reactor meltdown, this model has obvious limitations. In this way the students' insights and experiences with calculations of risk can be connected to important societal discussions. In that sense "the terrible small numbers" is exemplary for the dialogical and critical approach to mathematical teaching and for illustrating that the students' inquiry-based activities play a crucial role in this approach.

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