Proofs for "Bounding Multiple Gaussians Uncertainty with Application to object tracking"

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Theorem 1. *If $H'(z)$ is a wave function in the form of a weighted sum of M Gaussian functions, then the following inequality holds with $M' = \binom{M}{2} + M$*

$$\frac{\hbar}{2} \leq \sigma_z \cdot \sigma_k \leq \sqrt{M'\xi} \cdot \frac{\hbar}{2} \cdot \sqrt{\frac{\varpi_{max}}{\varpi_{min}}} \tag{1}$$

*where σ_z^2, σ_k^2 are respectively the variances of the position and the momentum of an object, ϖ_{max} and ϖ_{min} are respectively the maximum and minimum values among all the $\{\varpi_i\}$'s, $\hbar$ is the Planck's constant and ξ is the **tolerance factor**.*

Proof. The proof is provided in the main body of the submission. □

Theorem 2. *If two Gaussian mixture model is represented by $f(z, \mu_1, \sigma_1, \mu_2, \sigma_2)$ as:*

$$f(z, \mu_1, \sigma_1, \mu_2, \sigma_2) = w_1 G_1(z, \mu_1, \sigma_1) + w_2 G_2(z, \mu_2, \sigma_2)$$

then, we have:

$$f_{\sigma_z^2}(z, \mu_1, \sigma_1, \mu_2, \sigma_2) = f_{\sigma_z^2}(z, 0, \sigma_1, 0, \sigma_2) + \zeta \tag{2}$$

where $f_{\sigma_z^2}$ represents the variance of $f()$, with $\zeta \geq 0$ and $1 \geq \mu_1, \mu_2 \geq 0$.

Proof. Based on the definition of variance, we have:

$$\begin{aligned}
f_{\sigma_z^2} &= \int z^2(w_1G_1 + w_2G_2)dz - (\int z(w_qG_1 + w_2G_2)dz)^2 \\
&= w_1 \int z^2 G_1 dz - w_1(\int zG_1 dz)^2 + w_2 \int z^2 G_2 dz - \\
&\quad w_2(\int zG_2 dz)^2 + w_1(\int zG_1 dz)^2 - w_1^2(\int zG_1 dz)^2 \\
&\quad + w_2(\int zG_2 dz)^2 - w_2^2(\int zG_2 dz)^2 \\
&\quad -2w_1w_2\mu_1\mu_2
\end{aligned} \tag{3}$$

According to the definition of the variance, we have:

$$\begin{aligned} f_{\sigma_z^2} &= w_1\sigma_1^2 + w_2\sigma_2^2 + (w_1 - w_1^2)\mu_1^2 + (w_2 - w_2^2)\mu_2^2 \\ &\quad -2w_1w_2\mu_1\mu_2 \\ &= w_1\sigma_1^2 + w_2\sigma_2^2 + w_1w_2\mu_1^2 + w_1w_2\mu_2^2 \\ &\quad -2w_1w_2\mu_1\mu_2 \end{aligned} \tag{4}$$

As $w_1 + w_2 \leq 1$, we have:

$$w_1w_2 \leq \frac{1}{4}$$

so

$$f_{\sigma_z^2} = w_1\sigma_1^2 + w_2\sigma_2^2 + \frac{1}{4}(\mu_1 - \mu_2)^2 \tag{5}$$

since $f_{\sigma^2}(z, 0, \sigma1, 0, \sigma2) = w_1\sigma_1^2 + w_2\sigma_2^2$, and we have:

$$f_{\sigma_z^2}(z, \mu_1, \sigma_1, \mu_2, \sigma_2) = f_{\sigma_z^2}(z, 0, \sigma_1, 0, \sigma_2) + \zeta \tag{6}$$

where $\zeta \geq 0$ is much smaller than 0.25 in the case of $1 \geq \mu_1, \mu_2 \geq 0$. In our application, μ is calculated based on ℓ^t, which is first normalized by the size of the searching window(L) as $\frac{\ell^t}{L} + 0.5$. L is the $\sqrt{2}$ times of the size of the tracked object.
Similarly, Theorem 2 can be extended to the case of three Gaussian models with positive weights as shown in Definition 2, and in this case, ζ is much smaller than $\frac{3}{4}$.If the sum of weights is not 1, but the inequality in Theorem 2 is still valid only by changing ζ. In order to prove the uncertainty principle, we need to prove the following two assertions.

Assertion 1:
We discuss about the property of a wave function in the form of a Multiple Gaussian model, when the mean μ is non-zero. According to the property of Fourier transformation, we have:

$$Fourier(f(x \pm x_0)) = e^{\pm jwx_0}Fourier(f(x)) \tag{7}$$

where $Fourier()$ is the Fourier transformation function. For any $f()$, the above equation is correct. In the frequency domain, when $z \to z - \mu$, we have: $H'(z) = \sum_{i=1}^{M}\sqrt{w_i}(\frac{\varpi_i}{\pi})^{\frac{1}{4}}e^{\frac{-\varpi_i(z-\mu_i)^2}{2}}$. As $|e^{-jk\mu_i}| = 1$, we have:

$$\phi(k) = |\phi'(k)|^2|e^{-jk\mu_i}|^2 = |\phi'(k)|^2 \tag{8}$$

Thus we obtain the conclusion that σ_k^2 remains unchanged when the mean is shifted.

Assertion 2:
From Theorem 2, we can know that ζ is a small value when $M \leq 3$, if we can neglect, then from Theorem 2 and Modification 1, we derive the following similar result as the proof of Theorem 1:

$$\sigma_z^{'2} \cdot \sigma_k^{'2} \approx \sigma_z^2 \cdot \sigma_k^2 \leq \frac{h^2}{4}M'\frac{\varpi_{max}}{\varpi_{min}} \tag{9}$$

where ${\sigma'}_z^2, {\sigma'}_k^2$ are the corresponding variances when μ is nonzero for a multiple Gaussian model. By using the uncertainty principle, we can have:

$$\frac{h^2}{4} \leq {\sigma'}_z^2 \cdot {\sigma'}_k^2 \tag{10}$$

Based on the two assertions and Theorem 2, we can have the same result as that of Theorem 1 when μ the mean is non-zero for a multiple Gaussian model.
Remark 1-1: we can further discuss about the bound when the values in $W_i' \Omega_i'$ defined in the proof Theorem 1 is very small, even it happens very rare.
If $W_i' \Omega_i' \leq \zeta$ is satisfied, we can easily obtain the following conclusion:

$${\sigma'}_z^2 \cdot {\sigma'}_k^2 \approx \sigma_z^2 \cdot \sigma_k^2 \leq \frac{h^2}{4} \zeta \sum_i W_1'' \Xi_1' (M' + 1) \tag{11}$$

and we have:

$${\sigma'}_z^2 \cdot {\sigma'}_k^2 \approx \sigma_z^2 \cdot \sigma_k^2 \leq \frac{h^2}{4} \zeta \varpi'_{max} (M' + 1) \tag{12}$$

As the values in $W_i' \Omega_i'$ are quite small, and also M' can be chosen to be a small value, thus the bound can be tight.

□