Appendix for Implied Volatility And Skewness Surface

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A *Proposition 1*

Our candidate SDF is, for given ν,

$$M_t = \exp\left(-\nu\left(\Delta\right)\varepsilon_t + \Psi\left(\nu\left(\Delta\right)\right)\right),$$

where Ψ is the log-cumulant function of ε,

$$\Psi\left(u\right) = 2u\frac{\sqrt{h\left(\Delta\right)}}{\alpha\left(\Delta\right)} - \frac{4}{\alpha\left(\Delta\right)^2}\ln\left[1+\frac{1}{2}u\alpha\left(\Delta\right)\sqrt{h\left(\Delta\right)}\right].$$

Following CEFJ, this SDF defines an Equivalent Martingale Measure [EMM] if and only if

$$\Psi\left(\nu\left(\Delta\right)-1\right) - \Psi\left(\nu\left(\Delta\right)\right) - \Psi\left(-1\right) + \left(\mu - r_0\right)\Delta = 0,$$

which has the following unique solution for $\nu\left(\Delta\right)$,

$$\nu\left(\Delta\right) = -\frac{2}{\alpha\left(\Delta\right)\sqrt{h\left(\Delta\right)}} + \frac{g\left(\Delta\right)}{g\left(\Delta\right)-1},$$

where

$$g\left(\Delta\right) = \exp\left(-\frac{(\mu-r_0)\Delta}{4}\alpha\left(\Delta\right)^2 + \frac{\alpha\left(\Delta\right)\sqrt{h\left(\Delta\right)}}{2}\right).$$

Proposition 2 of CEFJ establishes sufficient conditions on Ψ for the solution to be unique.

B *Limit of Risk-Neutral Volatility*

Define

$$\Pi_0 \equiv (\mu - r)$$
$$\beta(\Delta) \equiv \alpha(\Delta)\frac{\sqrt{\Delta}}{2}$$
$$\sigma^*(\Delta) \equiv \sqrt{h^*(\Delta)}/\sqrt{\Delta},$$

and note that the drift correction term can be written as

$$2\frac{\sqrt{h^*\left(\Delta\right)} - \sqrt{h\left(\Delta\right)}}{\alpha\left(\Delta\right)} = \frac{\sigma^*\left(\Delta\right) - \sigma}{\beta(\Delta)}\Delta. \tag{1}$$

We first study the limit of the numerator as skewness tends to zero. Using the definitions above we have (see Proposition 2)

$$\sigma^*\left(\Delta\right) = \frac{g(\beta\left(\Delta\right)) - 1}{\beta(\Delta)g(\beta(\Delta))} \tag{2}$$

where, with a slight abuse of notation,

$$g(\beta(\Delta)) \equiv \exp(-\Pi_0\beta(\Delta)^2 + \beta(\Delta)\sigma), \tag{3}$$

which leads to an indeterminacy when skewness tends to zero. We use the first order expansion of the exponential function, $\exp(x) = 1 + x + x\theta(x)$ where $\theta(x)$ tends to zero when x tends to zero. Substituting in Equation 2 leads to, after some simplification,

$$\sigma^*\left(\Delta\right) = \frac{-\Pi_0\beta\left(\Delta\right) + \sigma + \theta\left(\beta\left(\Delta\right)\right)}{1 - \Pi_0\beta\left(\Delta\right)^2 + \beta\left(\Delta\right)\sigma + \beta\left(\Delta\right)\theta\left(\beta\left(\Delta\right)\right)},$$

and taking the limit shows that $\sigma^*\left(\Delta\right) \to \sigma$ when $\beta\left(\Delta\right) \to 0$.

Note then that the limit of 1 leads to an indeterminacy. We will again apply a Taylor expansion but, first, we compute the first order derivative of (2) with respect to $\beta(\Delta)$ using Equation (3) to compute the derivative of $g(\beta(\Delta))$ which leads to

$$\frac{d\sigma^*(\beta(\Delta))}{d\beta(\Delta)} = \frac{1 - g(\beta(\Delta)) + \beta(\Delta)(\sigma - 2\Pi_0\beta(\Delta))}{\beta(\Delta)^2 g(\beta(\Delta))},$$

where again we face an indeterminacy. We use a second-order expansion of $g(\beta(\Delta))$

$$g(\beta(\Delta)) = g(0) + \beta(\Delta) g'(0) + \frac{1}{2} g''(0)\beta(\Delta)^2 + \beta(\Delta)^2 \theta(\beta(\Delta)),$$

where $\theta(\beta(\Delta))$ tends to zero when $\beta(\Delta)$ tends to zero. Substituting these results in a first-order expansion for $\sigma^*(\beta(\Delta))$,

$$\sigma^*(\beta(\Delta)) = \sigma^*(0) + \frac{d\sigma^*(\beta(\Delta))}{d\beta(\Delta)}(0)\,\beta(\Delta) + \beta(\Delta)\theta(\beta(\Delta)),$$

leads to

$$\frac{\sigma^*(\Delta) - \sigma}{\beta(\Delta)} = -\left(\Pi_0 + \frac{\sigma^2}{2}\right) + \theta(\beta(\Delta)),$$

which, in the limit, delivers the desired result. Note that we then have

$$\mu\Delta + 2\frac{\sqrt{h^*(\Delta)} - \sqrt{h(\Delta)}}{\alpha(\Delta} = \left(r - \frac{\sigma^2}{2}\right)\Delta + \Delta\theta(\beta(\Delta)).$$

and, finally, that if we substitute the second-order expansion for $g(\Delta)$ in the solution for ν, we get

$$\nu(\Delta) \to \frac{\mu - r + \frac{\sigma^2}{2}}{\sigma^2} = \frac{\mu - r}{\sigma^2} + \frac{1}{2},$$

C Taylor Expansion of the Price of Risk

We want to show that,

$$\nu(\beta) \approx \frac{\mu - r}{\sigma^2} + \frac{1}{2} + \frac{(\mu - r)^2 + \frac{\sigma^4}{12}}{\sigma^3}\beta$$

where

$$\begin{aligned} \nu(\beta) &= -\frac{1}{\beta\sigma} + \frac{g(\beta)}{g(\beta) - 1} \\ g(\beta) &= \exp(-(\mu - r)\beta^2 + \beta\sigma). \end{aligned}$$

Recall that $\nu(0) = (\mu - r)/\sigma^2 + \frac{1}{2}$ and note that

$$\begin{aligned} \nu'(\beta) &= \frac{1}{\beta^2\sigma} - \frac{g'(\beta)}{(g(\beta) - 1)^2} \\ g'(\beta) &= (-2(\mu - r)\beta + \sigma)\, g(\beta), \end{aligned}$$

We evaluate the limit of this derivative as $\beta \to 0$ using, as above, the second-order expansion of $g(\beta)$. After tedious but straightforward computations, the result is

$$\begin{aligned} \nu'(0) &= \frac{(\mu - r)^2 + \frac{\sigma^4}{4} - 2(\mu - r)\sigma^2 + \frac{\sigma^4}{3} - (\mu - r)\sigma^2 + 2(\mu - r)\sigma^2 + (\mu - r)\sigma^2 - \frac{\sigma^4}{2}}{\sigma^3} \\ &= \frac{(\mu - r)^2 + \frac{\sigma^4}{12}}{\sigma^3}. \end{aligned}$$

D Proposition 2

From CEFJ, the logarithm risk-neutral of the risk-neutral Moment Generating Function is

$$\begin{aligned} \Psi^{Q*}(u) &= -u\Psi'(\nu(\Delta)) + \Psi(\nu(\Delta) + u) - \Psi(\nu(\Delta)) \\ &= 2u\frac{\sqrt{h^*(\Delta)}}{\alpha(\Delta)} - \frac{4}{\alpha(\Delta)^2}\ln\left[1 + \frac{1}{2}u\alpha(\Delta)\sqrt{h^*(\Delta)}\right], \end{aligned}$$

implying that

$$\varepsilon^*_{t+\Delta} = \frac{\sqrt{h(\Delta)}}{\sqrt{h^*(\Delta)}}\varepsilon_{t+\Delta} + \nu(\Delta)\sqrt{h(\Delta)}.$$

The HG model can then be written as

$$\ln(S_{t+\Delta}/S_t) = r_0\Delta - \gamma^*(\Delta) + \sqrt{h^*(\Delta)}\varepsilon^*_{t+\Delta},$$

where

$$\gamma^*(\Delta) = \Psi^{Q*}(-1) = \text{-}2\frac{\sqrt{h^*(\Delta)}}{\alpha(\Delta)} - \frac{4}{\alpha(\Delta)^2}\ln\left[1 - \frac{1}{2}\alpha(\Delta)\sqrt{h^*(\Delta)}\right],$$

and with

$$\sqrt{h^*(\Delta)} = \frac{2(g(\Delta)-1)}{\alpha(\Delta)g(\Delta)}.$$

Substituting back in the equation for returns under the risk-neutral measure, and simplifying, yields the results.

E Greeks

For notational simplicity we introduce $a \equiv H/\beta(\Delta)^2$. We begin with the sensitivity to changes in the underlying stock price. The HG option price is homogenous of degree one in stock price and strike. Then the standard result holds and the option delta is simply

$$\frac{\partial C_t}{\partial S_t} = C_{1,t}, \tag{4}$$

which depends on skewness. Next, the sensitivity of the option's delta with respect to the stock price is

$$\frac{\partial^2 C_t}{\partial S_t^2} = \frac{e^{-(d_2+r_f H)}d_2^{a-1}}{|\beta|\sigma^*\Gamma(a)}\frac{K}{S_t^2}, \tag{5}$$

which also depends on skewness and moneyness. The sensitivity of option prices to changes in the underlying risk-neutral volatility is

$$\frac{\partial C_t}{\partial \sigma_t^*} = \frac{|\beta|\sigma^* e^{(-r_f H)}K}{\sigma^*(1-\beta\sigma^*)}\frac{e^{-d_2}d_2^a}{\Gamma(a)}, \tag{6}$$

and, finally, the sensitivity of option prices to changes in the skewness of returns is given by

$$\begin{aligned}\frac{\partial C_t}{\partial \beta} &= -\frac{2a}{\beta}\left[(\ln(d_2) - \Psi(a))\,C_t - Ke^{(-r_f H)}P(a,d_2)\ln(1-\beta\sigma)\right] \\ &+ \frac{2a}{\beta}\Gamma(a)d_2^a S_t(1-\beta\sigma)^a{}_2\bar{F}_2(a,a;a+1,a+1;-d_1) \\ &- \frac{2a}{\beta}\Gamma(a)d_2^a Ke_2^{(-r_f H)}\bar{F}_2(a,a;a+1,a+1;-d_2) \\ &- Ke^{(-r_f H)}\frac{\sigma^*}{1-\beta\sigma^*}\frac{e^{-d_2}d_2^a}{\Gamma(a)},\end{aligned} \tag{7}$$

where

$$\Psi(a,z) = P(a,z)\ln(z) - \Gamma(a)\,z^a{}_2\bar{F}_2(a,a;a+1,a+1;-z),$$

and where ${}_2\bar{F}_2(\cdot)$ is the regularized hypergeometric function.

F Proposition 3

A no-arbitrage price of a European call option with strike price K and maturity T can be obtained from the computation of the discounted expectation of the terminal payoff under the risk-neutral measure. That is,

$$\begin{aligned}C_t(K,M) &= E^Q[\max(S_{t+T}-K,0)] \\ C_t &= \exp(-rT)S_t E^Q\left[\exp(R_{t,M})\,1_{[R_{t,M}>\ln(K/S_t)]}\right] - \exp(-r_0 T)\,KP^Q[R_{t,M} > \ln(K/S_t)].\end{aligned}$$

We can compute $P^Q[R_{t,M} > \ln(K/S_t)]$ from the distribution function of a gamma variable. Note first that

$$P^Q[R_{t,M} > \ln(K/S_t)] = P^Q\left[\frac{\beta(\Delta)}{\sqrt{\Delta M}}y^*_{t,M} > \frac{\ln(K/S_t) - \mu^*(\Delta)M\Delta}{\sqrt{\Delta M}\sigma^*(\Delta)} + \frac{\sqrt{\Delta M}}{\beta(\Delta)}\right],$$

where we define

$$\frac{2\sqrt{M}}{\alpha(\Delta)}\left(\varepsilon^*_{t,M}+\frac{2\sqrt{M}}{\alpha(\Delta)}\right)=y^*_{t,M}\sim^Q\Gamma\left(\frac{4M}{\alpha(\Delta)^2},1\right),$$

based on the characterization of the standardized Gamma distribution given in Equation **??**. If $\alpha(\Delta)>0$,

$$P^Q[R_{t,M}>\ln(K/S_t)]==\frac{\Gamma\left(\frac{T}{\beta^2(\Delta)};\frac{T}{\beta^2(\Delta)}+\frac{\ln(K/S_t)-\mu^*(\Delta)T}{\beta(\Delta)\sigma^*(\Delta)}\right)}{\Gamma\left(\frac{T}{\beta^2(\Delta)}\right)},$$

where $\Gamma(a,x)$ is the upper incomplete gamma function[1] and if $\alpha(\Delta)<0$,

$$\begin{aligned}P^Q[R_{t,M}>\ln(K/S_t)]&==\frac{\gamma\left(\frac{T}{\beta(\Delta)^2};\frac{T}{\beta(\Delta)^2}+\frac{\ln(K/S_t)-\mu^*T}{\beta(\Delta)\sigma^*(\Delta)}\right)}{\Gamma\left(\frac{T}{\beta(\Delta)^2}\right)}\\&=1-\frac{\Gamma\left(\frac{T}{\beta(\Delta)^2};\frac{T}{\beta(\Delta)^2}+\frac{\ln(K/S_t)-\mu^*T}{\beta(\Delta)\sigma^*(\Delta)}\right)}{\Gamma\left(\frac{T}{\beta(\Delta)^2}\right)}.\end{aligned}$$

Similarly,

$$\begin{aligned}&E^Q\left[\exp\left(R_{t,M}\right)1_{[R_{t,M}>\ln(K/S_t)]}\right]\\=\;&\exp\left(\mu^*(\Delta)M\Delta-\frac{\sigma^*(\Delta)M\Delta}{\beta(\Delta)}\right)E^Q\left[\exp\left(\sigma^*(\Delta)\beta(\Delta)y^*_{t,M}\right)1_{\left[\frac{\beta(\Delta)}{\sqrt{\Delta M}}y^*_{t,M}>\kappa\right]}\right]\end{aligned}$$

where we use

$$\kappa=\frac{\ln(K/S_t)-\mu^*(\Delta)M\Delta}{\sqrt{\Delta M}\sigma^*(\Delta)}+\frac{\sqrt{\Delta M}}{\beta(\Delta)}.$$

Then, if $\alpha(\Delta)>0$, and using that $y^*_{t,M}$ has a standard gamma distribution with parameter $\frac{M\Delta}{\beta(\Delta)^2}$, we have

$$\begin{aligned}&E^Q\left[\exp\left(\sigma^*(\Delta)\beta(\Delta)y^*_{t,M}\right)1_{\left[y^*_{t,M}>\frac{\sqrt{\Delta M}\kappa}{\beta(\Delta)}\right]}\right]\\=\;&\int_{(1-\sigma^*(\Delta)\beta(\Delta))\frac{\sqrt{\Delta M}\kappa}{\beta(\Delta)}}^{\infty}\exp\left(-z^*_{t,M}\right)\frac{\left(z^*_{t,M}\right)^{\frac{M\Delta}{\beta(\Delta)^2}-1}}{\left(1-\sigma^*(\Delta)\beta(\Delta)\right)^{\frac{M\Delta}{\beta(\Delta)^2}}\Gamma\left(\frac{M\Delta}{\beta(\Delta)^2}\right)}dz^*_{t,M}\\=\;&\frac{\Gamma\left(\frac{M\Delta}{\beta(\Delta)^2};\left(\frac{M\Delta}{\beta(\Delta)^2}+\frac{\ln(K/S_t)-\mu^*(\Delta)\Delta M}{\beta(\Delta)\sigma^*(\Delta)}\right)\left(1-\sigma^*(\Delta)\beta(\Delta)\right)\right)}{\Gamma\left(\frac{M\Delta}{\beta(\Delta)^2}\right)\left(1-\sigma^*(\Delta)\beta(\Delta)\right)^{\frac{M\Delta}{\beta(\Delta)^2}}}\end{aligned}$$

and, using the change of variables $(1-\sigma^*(\Delta)\beta(\Delta))y^*_{t,M}=z^*_{t,M}$, it follows that

$$\begin{aligned}&E^Q\left[\exp\left(R_{t,M}\right)1_{[R_{t,M}>\ln(K/S_t)]}\right]\\=\;&\exp\left(\left(\mu^*(\Delta)-\frac{\sigma^*(\Delta)}{\beta(\Delta)}\right)T\right)\frac{\Gamma\left(\frac{T}{\beta(\Delta)^2};\left(\frac{T}{\beta(\Delta)^2}+\frac{\ln(K/S_t)-\mu^*(\Delta)T}{\beta(\Delta)\sigma^*(\Delta)}\right)\left(1-\sigma^*(\Delta)\beta(\Delta)\right)\right)}{\Gamma\left(\frac{T}{\beta(\Delta)^2}\right)\left(1-\sigma^*(\Delta)\beta(\Delta)\right)^{\frac{T}{\beta(\Delta)^2}}}.\end{aligned}$$

If, however, $\alpha(\Delta)<0$ then

$$\begin{aligned}&E^Q\left[\exp\left(\sigma^*(\Delta)\beta(\Delta)y^*_{t,M}\right)1_{\left[\frac{\beta(\Delta)}{\sqrt{\Delta M}}y^*_{t,M}>\kappa\right]}\right]\\=\;&\frac{\gamma\left(\frac{M\Delta}{\beta(\Delta)^2};\left(\frac{M\Delta}{\beta(\Delta)^2}+\frac{\ln(K/S_t)-\mu^*(\Delta)\Delta M}{\beta(\Delta)\sigma^*(\Delta)}\right)\left(1-\sigma^*(\Delta)\beta(\Delta)\right)\right)}{\Gamma\left(\frac{M\Delta}{\beta(\Delta)^2}\right)\left(1-\sigma^*(\Delta)\beta(\Delta)\right)^{\frac{M\Delta}{\beta(\Delta)^2}}},\end{aligned}$$

[1]The upper incomplete gamma function is defined as $\Gamma(a,x)=\int_x^\infty t^{a-1}e^{-t}dt$ while the lower incomplete gamma function is defined as $\gamma(a,x)=\int_0^x t^{a-1}e^{-t}dt$. Note that $\Gamma(a)=\Gamma(a,0)$ while $\gamma(a)=\gamma(a,\infty)$.

and then

$$
\begin{aligned}
& E^{Q}\left[\exp\left(R_{t,M}\right)1_{\left[R_{t,M}>\ln(K/S_t)\right]}\right] \\
= \; & \exp\left(\left(\mu^{*}\left(\Delta\right)-\frac{\sigma^{*}\left(\Delta\right)}{\beta\left(\Delta\right)}\right)T\right)\frac{\gamma\left(\frac{T}{\beta(\Delta)^2};\left(\frac{T}{\beta(\Delta)^2}+\frac{\ln(K/S_t)-\mu^{*}(\Delta)T}{\beta(\Delta)\sigma^{*}(\Delta)}\right)\left(1-\sigma^{*}\left(\Delta\right)\beta\left(\Delta\right)\right)\right)}{\Gamma\left(\frac{T}{\beta(\Delta)^2}\right)\left(1-\sigma^{*}\left(\Delta\right)\beta\left(\Delta\right)\right)^{\frac{T}{\beta(\Delta)^2}}}.
\end{aligned}
$$

G *Proposition 4*

Suppose that the underlying stock price evolution under the risk-neutral measure is given by

$$R_T=\left(r-\delta\right)T+\sigma\sqrt{T}y$$

where δ is a risk-adjustment factor, y is a random number with mean zero, variance 1, skewness, $\frac{2\beta}{\sqrt{T}}$ and kurtosis, λ_2. Suppose also that the probability density of y is described by the following Edgeworth series expansion around the standardized gamma distribution:

$$f\left(y\right)=g\left(y\right)+\frac{\lambda_2-\frac{6\beta^2}{T}}{4!}\frac{d^4g\left(y\right)}{dy^4},$$

where $g(y)$ is the standardized gamma density function given by

$$g(y)=\frac{\sqrt{T}z^{a-1}e^{-z}}{|\beta|\,\Gamma(a)}\quad if\quad \beta y>-\sqrt{T},$$

and where $z=\frac{\sqrt{T}}{\beta}y+a$. Imposing that gross stock returns are a martingale under the risk-neutral measure,

$$
\begin{aligned}
E_0^{Q}[\exp\left(R_T\right)] &= E_0^{Q}[\exp\left(\left(r-\delta\right)T+\sigma\sqrt{T}y\right)] \\
&= \exp\left(\left(r-\delta\right)T\right)\int\exp\left(\sigma\sqrt{T}y\right)\left[g\left(y\right)+\frac{\lambda_2-\frac{6\beta^2}{T}}{4!}\frac{d^4g\left(y\right)}{dy^4}\right]dy,
\end{aligned}
$$

leads to the required risk-adjustment,

$$\delta=\frac{1}{T}\ln\left[\exp\left(-\frac{\sigma T}{\beta}-a\ln\left(1-\beta\sigma\right)\right)+\frac{\lambda_2-\frac{6\beta^2}{T}}{4!}\int\exp\left(\sigma\sqrt{T}y\right)\frac{d^4g\left(y\right)}{dy^4}dy\right].$$

The price of a European call option is

$$c_0^{*}=e^{-rT}\int_{-d_2^{*}}^{\infty}\left(S_0\exp\left(\left(r-\delta\right)T+\sigma\sqrt{T}y\right)-K\right)f\left(y\right)dy$$

where

$$d_2^{*}=\frac{\ln\left(S_0/K\right)+\left(r-\delta\right)T}{\sigma\sqrt{T}}.$$

We have

$$c_0^{*}=e^{-rT}\left[S_0\exp\left(\left(r-\delta\right)T\right)\int_{-d_2^{*}}^{\infty}\exp\left(\sigma\sqrt{T}y\right)f\left(y\right)dy-K\int_{-d_2^{*}}^{\infty}f\left(y\right)dy\right].$$

For the first integral, we have

$$\int_{-d_2^{*}}^{\infty}\exp\left(\sigma\sqrt{T}y\right)f\left(y\right)dy=\int_{-d_2^{*}}^{\infty}\exp\left(\sigma\sqrt{T}y\right)g\left(y\right)dy+\kappa\int_{-d_2^{*}}^{\infty}\exp\left(\sigma\sqrt{T}y\right)\frac{d^4g\left(y\right)}{dy^4}dy$$

and for $\beta\leq 0$, say, and $d_1^{*}=\bar{d}_2\left(1-\sigma\beta\right)$ we have

$$\int_{-d_2^{*}}^{\infty}\exp\left(\sigma\sqrt{T}y\right)g\left(y\right)dy=\frac{e^{-\sigma\beta a}}{\left(1-\sigma\beta\right)^{a}}P\left(a,d_1^{*}\right)$$

while

$$\int_{-d_2^*}^{\infty} \exp\left(\sigma\sqrt{T}y\right) \frac{d^4 g(y)}{dy^4} dy$$

$$= a^2 e^{-\sigma\beta a} \left[\begin{array}{c} \frac{P\left(a-4,d_1^*\right)}{(1-\sigma\beta)^{a-4}} - 4\frac{P\left(a-3,d_1^*\right)}{(1-\sigma\beta)^{a-3}} \\ +6\frac{P\left(a-2,d_1^*\right)}{(1-\sigma\beta)^{a-2}} - 4\frac{P\left(a-1,d_1^*\right)}{(1-\sigma\beta)^{a-1}} + \frac{P\left(a,d_1^*\right)}{(1-\sigma\beta)^{a}} \end{array} \right].$$

Next, for the second integral above,

$$\int_{-d_2^*}^{\infty} f(y)\, dy = \int_{-d_2^*}^{\infty} g(y)\, dy + \kappa \int_{-d_2^*}^{\infty} \frac{d^4 g(y)}{dy^4} dy$$

with

$$\begin{aligned} \int_{-d_2^*}^{\infty} g(y)\, dy &= \int_0^{a-d_2^*\frac{\sqrt{T}}{\beta}} \frac{z^{a-1}e^{-z}}{\Gamma(a)} dz = P\left(a, \bar{d}_2\right) \\ \int_{-d_2^*}^{\infty} \frac{d^4 g(y)}{dy^4} dy &= a^2 \left[\begin{array}{c} P\left(a-4, \bar{d}_2\right) - 4P\left(a-3, \bar{d}_2\right) + \\ 6P\left(a-2, \bar{d}_2\right) - 4P\left(a-1, \bar{d}_2\right) + P\left(a, \bar{d}_2\right) \end{array} \right]. \end{aligned}$$

H Identifying Restriction on the P-HG

The equality of prices from the true model and the P-HG for at-the-money options implies that

$$\begin{aligned} P(a, d_1^*) - P(a, d_2^*) = {} & P(a, d_1) - \left(1 + T^2\sigma^4\kappa\right) P(a, d_2) \\ & + \kappa\frac{T^2\sigma}{\beta^3}\left[-h''(d_2) + \sigma\beta h'(d_2) - \sigma^2\beta^2 h(d_2)\right], \end{aligned}$$

while the equality of the first derivative of prices implies

$$\begin{aligned} P(a, d_2^*) + \frac{\sigma_{I0}\gamma_1}{\bar{\sigma}\sqrt{T}} \frac{d_2^*\beta h(d_2^*)}{(1-\beta\sigma_{0I})} = {} & \left(1 + T^2\sigma^4\kappa\right) P(a, d_2) \\ & + \kappa a^2 \left[h'''(d_2) + \sigma^3\beta^3 h(d_2)\right], \end{aligned}$$

and, finally, the equality of the second derivatives implies

$$\begin{aligned} & \frac{h(d_2^*)}{\sigma_{0I}} \left[1 + \frac{(2a - d_1^* - d_2^*)\,\beta\sigma_{I0}\gamma_1}{(1-\beta\sigma_{0I})\,\bar{\sigma}\sqrt{T}} + \frac{d_2^*\beta^3\sigma_{I0}^3\gamma_1^2}{(1-\beta\sigma_{0I})^2\,\bar{\sigma}^2 T} + \frac{2d_2^*\beta^2\sigma_{I0}^2\gamma_2}{(1-\beta\sigma_{0I})\,\bar{\sigma}^2 T}\right] = \\ & \left(1 + T^2\sigma^4\kappa\right)\frac{h(d_2)}{\sigma} + \frac{\kappa a^2}{\sigma}\left[h^{(4)}(d_2) + \sigma^3\beta.^3 h'(d_2)\right]. \end{aligned}$$

Then, linearizing the left sides of the equations around $\sigma_0 = \sigma$, $\gamma_1 = 0$ and $\gamma_2 = 0$, respectively, and the right side around $\kappa = 0$ leads to

$$\begin{aligned} \frac{\sigma_{I0} - \sigma}{\sigma} &= a^2(1-\sigma\beta)\left(\sigma^3\beta^3\frac{P(a,d_2)}{h(d_2)\,d_2} + \frac{(a-1)(a-2)}{d_2^3} - \frac{(a-1)(2+\sigma\beta)}{d_2^2} + \frac{1+\sigma\beta+\sigma^2\beta^2}{d_2}\right)\kappa \\ \gamma_1 &= -\frac{\bar{\sigma}\sqrt{T}(a-d_1)}{\beta\sigma_2}\frac{\sigma_{I0}-\sigma}{\sigma} + \frac{\bar{\sigma}\sqrt{T}a^2(1-\sigma\beta)}{d_2}\left[\frac{\beta^3\sigma^3 P(a,d_2)}{h(d_2)} + 2\beta^2\sigma^2 + \frac{h^{(3)}(d_2)}{\beta\sigma h(d_2)}\right]\kappa \\ \gamma_2 &= -\frac{\bar{\sigma}^2 T}{2\beta^2\sigma^2 d_2}\left(\frac{h'(d_2)(a-d_1)}{h(d_2)} - 1 + \sigma\beta\right)\frac{\sigma_{I0}-\sigma}{\sigma} \\ & \quad + -\frac{\bar{\sigma}\sqrt{T}(2a-d_1-d_2)}{2d_2\beta\sigma}\gamma_1 + \left(\sigma + \frac{h'(d_2)}{\beta h(d_2)}\right)\frac{\sigma(1-\sigma\beta)\,\bar{\sigma}^2 T^3}{2d_2\beta^2}\kappa \end{aligned}$$

where

$$d_2 = \frac{-a\ln(1-\sigma\beta)}{\sigma\beta},\ d_1 = d_2(1-\sigma\beta),\ a = \frac{T}{\beta^2},\ \kappa = \frac{\lambda_2 - \frac{6\beta^2}{T}}{4!}.$$