Interacting Information Cascades: Online Appendix

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Abstract

This online appendix collects additional examples and results, as well as omitted proofs.

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1 Introduction

This online appendix collects additional examples and results, as well as omitted proofs:

- Section 2 (p. 2) – Presents an extension concerning asymmetric priors.
- Section 3 (p. 9) – Provides the detailed proofs of Lemmas A2 and A3 and of Proposition 4, which were omitted from the Appendix of the main text.
- Section 4 (p. 21) – Presents an Illustrative Example illustrates the results of the main text and provides intuition. It also gives two examples that explore extensions of an interacting cascade: in one example there are two common players (p. 28) and in the other the identity of the common player is unknown (p. 29).

2 Asymmetric Priors

In this section we study the case where players have asymmetric priors regarding the true state. In the body of the paper we assume that the states are equally likely, i.e., $P(\mathcal{H}) = P(\mathcal{L}) = 1/2$, while here we assume that $P(\mathcal{H}) > 1/2 > P(\mathcal{L})$. We show that equilibrium has the same qualitative features whether priors symmetric or asymmetric: if a cascade develops in one group but not the other, then it spills over via the common player, and when no cascade has developed in either group, then the common player triggers the same cascade in both groups.

When priors are asymmetric, however, equilibrium is asymmetric, and this leads to an interesting difference between the two settings when cascades disagree. Suppose group A, say, is in a cascade on a, and group B is in a cascade on r. Then the common player follows her own signal. If she chooses r, then the cascade on a is *replaced* by a cascade on r, i.e., each player in group A moving after the common player chooses r, regardless of his own signal. The intuition for this result is that a cascade on r is highly informative as it requires overcoming the favorable prior on state $\mathcal{H}$. Thus, when the common player chooses r, the existing cascade on a is not merely broken, but rather it is replaced altogether. If, alternatively, the common player chooses a, then this "re-initializes" group B, with player $k + 1^B$ and her successors in B playing as if $k + 1^B$ were the first player in a basic cascade.

Throughout this section we maintain two assumptions. Assumption OA1 requires that the signal precision exceeds the prior belief of the $\mathcal{H}$ state. It guarantees that, absent any other information, a player follows her own signal (and, in particular, chooses r following an L signal).

Assumption OA1: We have $p > P(\mathcal{H})$.

The second assumption is that the initial prior $P(\mathcal{H})$ is not too favorable for the $\mathcal{H}$ state. Recall that $\lambda = 2p(1-p)$. Let $q = P(\mathcal{H})$ and let

$$\alpha_k = \frac{\lambda^{\frac{k-1}{2}}}{2} \text{ and } \beta_k = \frac{1 - \frac{\lambda}{2}^{\frac{k-1}{2}}}{1 - \frac{\lambda}{2}}.$$

Assumption OA2 ensures that a cascade on r spills over from one group to another when the second group is not in a cascade.[1]

Assumption OA2: The prior q satisfies

$$q < \frac{\alpha_k + p\beta_k}{2\alpha_k + \beta_k}.$$

It is easy to verify that $p > 1/2$ implies $\frac{\alpha_k + p\beta_k}{2\alpha_k + \beta_k} > \frac{1}{2}$, and thus for every k there is a non-empty set of values $q \in (\frac{1}{2}, \frac{\alpha_k + p\beta_k}{2\alpha_k + \beta_k}]$ for which OA2 holds. Furthermore, $p > q$ by OA1 and $\lim_{k\to\infty} \alpha_k = 0$ implies that OA2 is always satisfied for k sufficiently large.

We now state our main result, Proposition OA1, for asymmetric priors. The statements of Proposition OA1 and Proposition 3 (in the main text) differ for only for POA1.3, which deals with opposing cascades. It is worth noting, however, that the conditions under which cascades form, as described in Lemma OA3 below, differ when priors are asymmetric.

Proposition OA1. (Asymmetric Priors): *Assume OA1 and OA2. In equilibrium, if prior to the common player:*

$POA1.1$: *Both groups are in cascades on a (r), then each cascade continues, i.e., the common player and all subsequent players in both groups choose a (r).*

$POA1.2$: *One group is in a cascade on a (r) and the other group is not in a cascade, then the cascade on a (r) "spills over" to the other group, i.e., the common player and all the subsequent players in both groups choose a (r).*

[1] See the proof of Lemma OA5.1.

POA1.3 : One group, say A, is in a cascade on a and the other group, B, is in a cascade on r then (i) the common player follows her own signal, (ii) if the common player chooses a, then the cascade in A continues (i.e., all subsequent players in A choose a) and the cascade in B ends (i.e., player $k+1^B$ is the first player in a basic cascade), and (iii) if the common player chooses r, then all subsequent players in both groups choose r, i.e., the cascade in A on a is replaced by a cascade on r.

POA3.4 : Neither group is in a cascade, then the common player "triggers" two cascades, i.e., the common player follows her own signal and all subsequent players in both groups make the same decision as the common player.

Proof: Follows directly from Lemmas OA1 through OA3, which are established below. □

With asymmetric priors, it is necessary to redefine the notions of a balanced and an unbalanced history. A history $\bar{d}_i = (d_1, \ldots, d_i)$ is **balanced** if (i) $d_j = r$ for $j \leq i$ and j odd and (ii) $d_j = a$ for $j \leq i$ and j even; it is **unbalanced** otherwise. The empty history is trivially balanced, as is the singleton (r) history. A history $\bar{d}_i = (d_1, \ldots, d_i)$ is **unbalanced on *a*** if at some point in the profile it switches from a balanced profile to a profile of only a, i.e., if there is an odd $j \leq i$ such that (i) $(d_1, \ldots, d_{j-1})$ is balanced and (ii) $d_j = \cdots = d_i = a$. The singleton (a) history is trivially unbalanced on a. A history $\bar{d}_i = (d_1, \ldots, d_i)$ is **unbalanced on *r*** at some point in the profile it switches from a balanced profile to a profile of only r and there are at least two such r decisions, i.e., if there is an odd $j \leq i-1$ such that (i) $(d_1, \ldots, d_{j-1})$ is balanced, and (ii) $d_j = \cdots = d_i = r$. For example, the history (r, a, r) is balanced, the history (r, a, a) is unbalanced on a, and the history (r, a, r, r) is unbalanced on r. Let D_i^b, D_i^a, and D_i^r be, respectively, the set of i-length histories that are balanced, unbalanced on a, and unbalanced on r.

Lemma OA1 characterizes equilibrium for players who move prior to the common player. It shows that each player follows her signal if the profile of prior decisions is balanced, she chooses a if the profile of prior decisions is unbalanced on a, and she chooses r if the profile of prior decisions is unbalanced on r. The lemma shows that a cascade on a is "less informative" than a cascade on r – e.g., a cascade on a starts after player 1 chooses a, which occurs after a single H signal, while a cascade on r starts after players 1 and 2 choose r, which occurs after two L signals. Note that having redefined the notion of a balanced and unbalanced decision profiles, Table OA1(a) in Lemma OA1 is identical to Table A1(a) in the Appendix.

Lemma OA1. Equilibrium Play for Predecessors of the Common Player.

Let $i < k$ and $g \in \{A, B\}$. In equilibrium, $\bar{d}^g_{i-1}$ belongs to a row in Table OA1(a) and player i^g's equilibrium strategy $\sigma^{g*}_i(\bar{d}^g_{i-1}, x^g_i)$ is given by the last two columns.

	$\sigma^{g*}_i(\bar{d}^g_{i-1}, x^g_i)$	
$\bar{d}^g_{i-1}$	H	L
D^a_{i-1}	a	a
D^b_{i-1}	a	r
D^r_{i-1}	r	r

(a) Player $i^g < k$

		$\sigma^*_k(\bar{d}^A_{k-1}, \bar{d}^B_{k-1}, x_k)$	
$\bar{d}^A_{k-1}$	$\bar{d}^B_{k-1}$	H	L
D^a_{k-1}	D^a_{k-1}	a	a
	D^b_{k-1}	a	a
	D^r_{k-1}	a	r
D^b_{k-1}	D^a_{k-1}	a	a
	D^b_{k-1}	a	r
	D^r_{k-1}	r	r
D^r_{k-1}	D^a_{k-1}	a	r
	D^b_{k-1}	r	r
	D^r_{k-1}	r	r

(b) Player k

Table OA1: Equilibrium Strategies of Players 1 through k

Proof: The proof is inductive. For player 1^g, the lemma is easily seen to hold. Suppose the lemma holds for players 1^g to $i-1^g$. We prove it holds for i^g. The induction hypothesis gives that $\bar{d}^g_{i-1}$ is in a row of the table.

If $\bar{d}^g_{i-1}$ is in D^a_{i-1}, then there is an odd j, where $j \leq i-1$, such that $(d_1, \ldots, d_{j-1})$ is balanced and $d_j = \cdots = d_{i-1} = a$. Player i^g infers that players 1^g through $j-1^g$ followed their own signals, with players in an odd position receiving a signal of L and in an even position receiving a signal of H. She infers as well, from $d^g_j = a$, that $x^g_j = H$. Hence player i^g's belief given her own signal x^g_i is

$$\phi^{g*}_i(\bar{d}^g_{i-1}, x^g_i) = \begin{cases} \frac{p^2 q}{p^2 q + (1-p)^2(1-q)} > \frac{1}{2} & \text{if } x^g_i = H \\ q & \text{if } x^g_i = L. \end{cases}$$

Thus $\sigma^{g*}_i(\bar{d}^g_{i-1}, x^g_i) = a$ for $x^g_i \in \{H, L\}$. Analogous arguments establish the result for $\bar{d}^g_{i-1} \in D^r_{i-1}$ or $\bar{d}^g_{i-1} \in D^b_{i-1}$. $\square$

The next lemma establishes that equilibrium play for the common player is the same whether priors are asymmetric or symmetric: The common player is in a cascade on a (r)

if either (i) both groups are in a cascade on a (r) or (ii) one group is in a cascade on a (r) and the other group is not in a cascade. Otherwise, the common player chooses a (r) given the signal H (L). Table OA1(b) describes precisely equilibrium play for the common player, and is identical to Table A1(b) in the Appendix.

Lemma OA2. Equilibrium play for the Common Player.
In equilibrium, $(\bar{d}^A_{k-1}, \bar{d}^B_{k-1})$ belongs to a row in Table OA1(b) and the common player's equilibrium strategy $\sigma^*_k(\bar{d}^A_{k-1}, \bar{d}^B_{k-1}, x_k)$ is given by the last two columns.

Proof: Suppose $\bar{d}^A_{k-1} \in D^a_{k-1}$ and $\bar{d}^B_{k-1} \in D^r_{k-1}$. Player i^g's belief given the history $(\bar{d}^A_{k-1}, \bar{d}^B_{k-1})$ and her own signal x_k is

$$\phi^*_k(\bar{d}^A_{k-1}, \bar{d}^B_{k-1}, x_k) = \begin{cases} q & \text{if } x_k = H \\ \frac{(1-p)^2 q}{(1-p)^2 q + p^2(1-q)} & \text{if } x_k = L. \end{cases}$$

Since $\phi^*_k(\bar{d}^A_{k-1}, \bar{d}^B_{k-1}, H) = q > 1/2$ then $\sigma^*_k(\bar{d}^A_{k-1}, \bar{d}^B_{k-1}, H) = a$. Furthermore, $p > q > 1/2$ implies $\phi^*_k(\bar{d}^A_{k-1}, \bar{d}^B_{k-1}, L) < 1/2$, and thus $\sigma^*_k(\bar{d}^A_{k-1}, \bar{d}^B_{k-1}, L) = r$. The other cases are established analogously, and are omitted. □

Lemma OA3 characterizes equilibrium play following the common player. Inspection of Table OA2(a), below, shows that player $k+1^g$ is certain to be in a cascade unless her group was in a cascade on r prior to the common player and the common player chose the opposing decision a, i.e., unless $\bar{d}^g_{k-1} \in D^r_{k-1}$ and $d_k = a$. In the later case she acts as the first player in a basic cascade.

Lemma OA3. Equilibrium After the Common Player.
Let $g \in \{A, B\}$.
$LOA3.1$: (Player $k+1^g$.) In equilibrium, $\bar{d}^g_k = (\bar{d}^g_{k-1}, d_k)$ belongs to a row in Table OA2(a) and player $k+1^g$'s equilibrium strategy $\sigma^{g*}_{k+1}(\bar{d}^g_k, x^g_{k+1})$ is given by the last two columns.
$LOA3.2$: (Player $k+2^g$.) In equilibrium, $\bar{d}^g_{k+1} = (\bar{d}^g_{k-1}, d_k, d^g_{k+1})$ belongs to a row in Table OA2(b) and player $k+2^g$'s equilibrium strategy $\sigma^{g*}_{k+2}(\bar{d}^g_{k+1}, x^g_{k+2})$ is given by the last two columns.
$LOA3.3$: (Subsequent players.) Let $i > k+2$. In equilibrium, $\bar{d}^g_{i-1} = (\bar{d}^g_{k-1}, d_k, d^g_{k+1}, \ldots, d^g_{i-1})$ belongs to a row in Table OA2(c) and player i^g's equilibrium strategy $\sigma^{g*}_i(\bar{d}^g_{i-1}, x^g_i)$ is given

by the last two columns.

$\bar{d}^g_{k-1}$	d_k	$\sigma^{g*}_{k+1}(\bar{d}^g_k, x^g_{k+1})$ H	L
D^a_{k-1}	a	a	a
	r	r	r
D^b_{k-1}	a	a	a
	r	r	r
D^r_{k-1}	a	a	r
	r	r	r

(a) Player $k+1^g$

$\bar{d}^g_{k-1}$	d_k	d^g_{k+1}	$\sigma^{g*}_{k+2}(\bar{d}^g_{k+1}, x^g_{k+2})$ H	L
D^a_{k-1}	a	a	a	a
	r	r	r	r
D^b_{k-1}	a	a	a	a
	r	r	r	r
D^r_{k-1}	a	a	a	a
	a	r	a	r
	r	r	r	r

(b) Player $k+2^g$

$\bar{d}^g_{k-1}$	d_k	$(d^g_{k+1}, \ldots, d^g_{i-1})$	$\sigma^{g*}_i(\bar{d}^g_{i-1}, x^g_i)$ H	L
D^a_{k-1}	a	$(a, \ldots, a)$	a	a
	r	$(r, \ldots, r)$	r	r
D^b_{k-1}	a	$(a, \ldots, a)$	a	a
	r	$(r, \ldots, r)$	r	r
D^r_{k-1}	a	a	a	a
	a	D^a_{i-k-1}	a	a
		D^b_{i-k-1}	a	r
		D^r_{i-k-1}	r	r
	r	$(r, \ldots, r)$	r	r

(c) Player $i^g > k+2^g$

Table OA2: Equilibrium Strategies of Players $k+1$ through N

Proof: We focus on the cases where equilibrium play differs from when priors are symmetric, as well as on the case where Assumption OA2 is required.

$LOA3.1$: Suppose that $\bar{d}^A_{k-1} \in D^a_{k-1}$ and $d_k = r$, i.e., consider row 2 of Table OA2(a). Then Lemma OA2, Table OA1(b), implies that $\bar{d}^B_{k-1} \in D^r_{k-1}$, i.e., group B was in a cascade on r prior to the common player's decision and $x_k = L$. Player $k+1^A$ infers, in addition, that the members of her group observed one more H signal than L signal before the cascade

began, while the members of group B have observed two more L signals than H signals before their cascade began. Hence,

$$\phi_{k+1}^{A*}(\bar{d}_{k-1}^{A}, r, x_{k+1}) = \begin{cases} \frac{(1-p)q}{(1-p)q+p(1-q)} & \text{if } x_{k+1}^{A} = H \\ \frac{(1-p)^3 q}{(1-p)^3 q+p^3(1-q)} & \text{if } x_{k+1}^{A} = L. \end{cases}$$

We have $\phi_{k+1}^{A*}(\bar{d}_{k-1}^{g}, r, L) < \phi_{k+1}^{A*}(\bar{d}_{k-1}^{g}, r, H) < 1/2$, where the second inequality holds since $p > q$ by Assumption OA1. Hence $\sigma_{k+1}^{A}(\bar{d}_{k-1}^{A}, r, x_{k+1}^{A}) = r$ for $x_{k+1}^{A} \in \{H, L\}$. This establishes row 2 of Table OA2(a).

Suppose that $\bar{d}_{k-1}^{A} \in D_{k-1}^{b}$ and $d_k = r$, i.e., consider row 4 of Table OA2(a). Then Lemma OA2, Table OA1(b), implies that either $\bar{d}_{k-1}^{B} \in D_{k-1}^{b}$ and $x_k = L$, or $\bar{d}_{k-1}^{B} \in D_{k-1}^{r}$. The probability that $\bar{d}_{k-1}^{B} \in D_{k-1}^{b}$, i.e., the probability of signal profile $(L, H, \ldots, L, H)$ of $k-1$ signals is $p^{\frac{k-1}{2}}(1-p)^{\frac{k-1}{2}}$, in each state. Thus the probability $\bar{d}_{k-1}^{B} \in D_{k-1}^{b}$ and $x_k = L$ is $p^{\frac{k-1}{2}}(1-p)^{\frac{k+1}{2}} = (1-p)\frac{\lambda}{2}^{\frac{k-1}{2}}$ in state $\mathcal{H}$ and $p^{\frac{k+1}{2}}(1-p)^{\frac{k-1}{2}} = p\frac{\lambda}{2}^{\frac{k-1}{2}}$ in state $\mathcal{L}$. The probability $\bar{d}_{k-1}^{B} \in D_{k-1}^{r}$ is the probability that the signal profile for group B is (L, L), or (L, H, L, L), or (L, H, L, H, L, L), etc. . In state $\mathcal{H}$ the probability of this event is

$$(1-p)^2(1 + (1-p)p + (1-p)^2p^2 + \ldots + (1-p)^{\frac{k-3}{2}}p^{\frac{k-3}{2}}) = (1-p)^2\frac{1-\frac{\lambda}{2}^{\frac{k-1}{2}}}{1-\frac{\lambda}{2}}.$$

In state $\mathcal{L}$, the probability of this event is

$$p^2(1 + (1-p)p + (1-p)^2p^2 + \ldots + (1-p)^{\frac{k-3}{2}}p^{\frac{k-3}{2}}) = p^2\frac{1-\frac{\lambda}{2}^{\frac{k-1}{2}}}{1-\frac{\lambda}{2}}.$$

Thus player $k+1^A$'s belief is $\phi_{k+1}^{A*}(\bar{d}_{k-1}^{A}, r, H)$, which equals

$$\frac{[(1-p)\frac{\lambda}{2}^{\frac{k-1}{2}} + (1-p)^2\frac{1-\frac{\lambda}{2}^{\frac{k-1}{2}}}{1-\frac{\lambda}{2}}]pP(\mathcal{H})}{[(1-p)\frac{\lambda}{2}^{\frac{k-1}{2}} + (1-p)^2\frac{1-\frac{\lambda}{2}^{\frac{k-1}{2}}}{1-\frac{\lambda}{2}}]pP(\mathcal{H}) + [p\frac{\lambda}{2}^{\frac{k-1}{2}} + p^2\frac{1-\frac{\lambda}{2}^{\frac{k-1}{2}}}{1-\frac{\lambda}{2}}](1-p)P(\mathcal{L})}$$

$$= \frac{[\alpha_k + (1-p)\beta_k]P(\mathcal{H})}{[\alpha_k + (1-p)\beta_k]P(\mathcal{H}) + [\alpha_k + p\beta_k]P(\mathcal{L})}.$$

We have $\phi_{k+1}^{A*}(\bar{d}_{k-1}^{A}, r, L) < \phi_{k+1}^{A*}(\bar{d}_{k-1}^{g}, r, H) < 1/2$, where the second inequality holds by Assumption OA2. Hence $\sigma_{k+1}^{A}(\bar{d}_{k-1}^{A}, r, x_{k+1}^{A}) = r$ for $x_{k+1}^{A} \in \{H, L\}$. This establishes row 4 of Table OA2(a).

Suppose that $\bar{d}_{k-1}^{A} \in D_{k-1}^{r}$ and $d_k = a$, i.e., consider row 5 of Table OA2(a). Then Lemma OA2, Table OA1(b), implies that $\bar{d}_{k-1}^{B} \in D_{k-1}^{a}$, i.e., group B was in a cascade on

a and $x_k = H$. Player $k+1^A$ infers, in addition, that the members of her own group have observed two more L signals than H signals before the cascade began, and the members of group B have observed one more H signal than L signal before the cascade began. Hence, prior to observing her own signal she assigns probability q to state $\mathcal{H}$. If $\bar{d}^A_{k-1} \in D^r_{k-1}$ and $d_k = a$, then player $k+1^A$'s belief is

$$\phi^{A*}_{k+1}(\bar{d}^A_{k-1}, a, x^A_{k+1}) = \begin{cases} \frac{pq}{pq+(1-p)(1-q)} & \text{if } x^A_{k+1} = H \\ \frac{(1-p)q}{(1-p)q+p(1-q)} & \text{if } x^A_{k+1} = L. \end{cases}$$

We have $\phi^{A*}_{k+1}(\bar{d}^A_{k-1}, a, H) > q > 1/2 > \phi^{A*}_{k+1}(\bar{d}^A_{k-1}, a, H)$ where the first inequality follows from $p > 1/2$ and last inequality follows from Assumption OA1. This establishes row 5 of Table OA2(a).

$LOA3.2$: Table OA2(b) is the same as Table A2(b) except for the removal of the row $\bar{d}^g_{k-1} \in D^a_{k-1}$, $d_k = r$, and $d^g_{k+1} = a$, since this case does not arise in equilibrium when priors are asymmetric. In particular, if a group is in a cascade on a and the common player chooses r, then the cascade in a is replaced by a cascade on r (as shown in the proof of $LOA3.1$). The detailed are analogous to those in the proof of $LOA3.1$.

$LOA3.3$: Table OA2(c) differs from Table A2(c) by the removal of rows 2, 3, and 4 in Table A2(c) that deal with the subcases of $\bar{d}^g_{k-1} \in D^a_{k-1}$, $d_k = r$, and $d^g_{k+1} = a$. Following the removal of these rows, the table is more compactly represented by combining the columns d^g_{k+1} and $(d^g_{k+2}, \ldots, d^g_{i-1})$. The details are are analogous to those in the proof of $LOA3.1$. □

3 Omitted Proofs

In this section, we provide the proofs of Lemmas A2 and A3 and of Proposition 4, which were omitted from the Appendix. We first develop notation and definitions.

An strategy profile σ is a list $(\sigma^A_1, \sigma^B_1, \ldots, \sigma^A_{k-1}, \sigma^B_{k-1}, \sigma_k, \sigma^A_{k+1}, \sigma^B_{k+1}, \ldots, \sigma^A_N, \sigma^B_N)$. Given a strategy profile σ, the probability that the decision profile of the first i players in group g is $\bar{d}^g_i = (\bar{d}^g_{i-1}, d^g_i) \in \{a, r\}^i$ when the true state is s is denoted by $P^g_\sigma(\bar{d}^g_i|s)$, and is defined recursively. For $g \in \{A, B\}$ and $i = 1$ we have

$$P^g_\sigma(\bar{d}^g_1|s) = \sum_{\{x|\sigma^g_1(x)=d^g_1\}} P(x|s).$$

For $g \in \{A, B\}$ and $i \in \{2, \ldots, N\} \backslash \{k\}$ we have

$$P_\sigma^g(\bar{d}_{i-1}^g, d_i^g|s) = P_\sigma^g(\bar{d}_{i-1}^g|s) \sum_{\{x|\sigma_i^g(\bar{d}_{i-1}^g, x) = d_i^g\}} P(x|s).$$

For group g and $i = k$ we have

$$P_\sigma^g(\bar{d}_{k-1}^g, d_k|s) = P_\sigma^g(\bar{d}_{k-1}^g|s) \sum_{\bar{d}_{k-1}^{-g} \in \{a,r\}^{k-1}} \left(P_\sigma^{-g}(\bar{d}_{k-1}^{-g}|s) \sum_{\{x|\sigma_k(\bar{d}_{k-1}^g, \bar{d}_{k-1}^{-g}, x) = d_k\}} P(x|s) \right),$$

where $-g$ denotes the other group. For a set $T \subset D_i^a \cup D_i^b \cup D_i^r$, we define $\text{Prob}(T|s) = \sum_{\bar{d}_i \in T} P_{\sigma^*}(\bar{d}_i|s)$, where σ^* is an equilibrium strategy profile. Since all equilibria have the same equilibrium path, $\text{Prob}(T|s)$ is invariant to the specific σ^* selected.

The next lemma will prove useful.

Lemma OA4. *Let $i < k$ and $g \in \{A, B\}$. If $\bar{d}_i^g$ is a history that results from equilibrium play, then $\bar{d}_i^g \in D_i^a \cup D_i^b \cup D_i^r$. In addition, we have*

$$\text{Prob}(D_i^b|s) = \begin{cases} \lambda^{\frac{i}{2}} & \textit{for } i \textit{ even} \\ \lambda^{\frac{i-1}{2}} & \textit{for } i \textit{ odd,} \end{cases}$$

$$\text{Prob}(D_i^a|s) = \begin{cases} P(H|s)^2 \Lambda(i) & \textit{for } i \textit{ even} \\ P(H|s)^2 \Lambda(i-1) & \textit{for } i \textit{ odd,} \end{cases}$$

$$\text{Prob}(D_i^r|s) = \begin{cases} P(L|s)^2 \Lambda(i) & \textit{for } i \textit{ even} \\ P(L|s)^2 \Lambda(i-1) & \textit{for } i \textit{ odd.} \end{cases}$$

Proof. Successive application of Lemma A1 gives that $\bar{d}_i^g \in D_i^a \cup D_i^b \cup D_i^r$. As to the probabilities, $(d_1, \ldots, d_i) \in D_i^b$ if and only if $d_1 \neq d_2, d_3 \neq d_4, \ldots, d_{i-1} \neq d_i$ when i is even. This occurs if and only if $x_1 \neq x_2, x_3 \neq x_4, \ldots, x_{i-1} \neq x_i$ by Lemma A1, an event of probability $\lambda^{\frac{i}{2}}$ in either state if i is even. The other probabilities are computed analogously. □

We restate Lemma A2 before we prove it.

Lemma A2. Equilibrium play for the Common Player.
In equilibrium, $(\bar{d}_{k-1}^A, \bar{d}_{k-1}^B)$ belongs to a row in Table A1(b) and the common player's equilibrium strategy $\sigma_k^*(\bar{d}_{k-1}^A, \bar{d}_{k-1}^B, x_k)$ is given by the last two columns.

		$\sigma_k^*(\bar{d}_{k-1}^A, \bar{d}_{k-1}^B, x_k)$	
$\bar{d}_{k-1}^A$	$\bar{d}_{k-1}^B$	H	L
D_{k-1}^a	D_{k-1}^a	a	a
	D_{k-1}^b	a	a
	D_{k-1}^r	a	r
D_{k-1}^b	D_{k-1}^a	a	a
	D_{k-1}^b	a	r
	D_{k-1}^r	r	r
D_{k-1}^r	D_{k-1}^a	a	r
	D_{k-1}^b	r	r
	D_{k-1}^r	r	r

Table A1(b): Equilibrium Strategies of Player k.

Proof. Let $(\bar{d}_{k-1}^A, \bar{d}_{k-1}^B) \in (D_{k-1}^b \cup D_{k-1}^a \cup D_{k-1}^r)^2$. By Lemma A1, we have $P_{\sigma^*}(\bar{d}_{k-1}^g|s) = (\lambda/2)^{\frac{k-1}{2}}$ if $\bar{d}_{k-1}^g \in D_{k-1}^b$. The lemma also gives $P_{\sigma^*}(\bar{d}_{k-1}^g|s) = (\lambda/2)^{\frac{\theta-1}{2}} p(H|s)^2$ if $\bar{d}_{k-1}^g \in D_{k-1}^a$, where θ is the smallest odd integer $j < i-1$ such that $d_j = d_{j+1} = a$, and $P_{\sigma^*}(\bar{d}_{k-1}^g|s) = (\lambda/2)^{\frac{\theta-1}{2}} p(L|s)^2$ if $\bar{d}_{k-1}^g \in D_{k-1}^r$, where θ is the smallest odd integer $j < i-1$ such that $d_j = d_{j+1} = r$. Thus, Bayes' rule gives that k's belief is

$$\phi_k^*(\bar{d}_{k-1}^A, \bar{d}_{k-1}^B, x_k) = \frac{P(\mathcal{H})P(x_k|\mathcal{H})P_{\sigma^*}(\bar{d}_{k-1}^A|\mathcal{H})P_{\sigma^*}(\bar{d}_{k-1}^B|\mathcal{H})}{\sum_{s\in\{\mathcal{H},\mathcal{L}\}} P(s)P(x_k|s)P_{\sigma^*}(\bar{d}_{k-1}^A|s)P_{\sigma^*}(\bar{d}_{k-1}^B|s)}.$$

This expression and a bit of algebra, yields Table OA3, a description of when $\phi_k^*(\cdot)$ is greater than, less than, or equal to $1/2$.

		x_k	
$\bar{d}^A_{k-1}$	$\bar{d}^B_{k-1}$	H	L
D^a_{k-1}	D^a_{k-1}	$\phi^*_k > \frac{1}{2}$	$\phi^*_k > \frac{1}{2}$
	D^b_{k-1}	$\phi^*_k > \frac{1}{2}$	$\phi^*_k > \frac{1}{2}$
	D^r_{k-1}	$\phi^*_k > \frac{1}{2}$	$\phi^*_k < \frac{1}{2}$
D^b_{k-1}	D^a_{k-1}	$\phi^*_k > \frac{1}{2}$	$\phi^*_k > \frac{1}{2}$
	D^b_{k-1}	$\phi^*_k > \frac{1}{2}$	$\phi^*_k < \frac{1}{2}$
	D^r_{k-1}	$\phi^*_k < \frac{1}{2}$	$\phi^*_k < \frac{1}{2}$
D^r_{k-1}	D^a_{k-1}	$\phi^*_k > \frac{1}{2}$	$\phi^*_k < \frac{1}{2}$
	D^b_{k-1}	$\phi^*_k < \frac{1}{2}$	$\phi^*_k < \frac{1}{2}$
	D^r_{k-1}	$\phi^*_k < \frac{1}{2}$	$\phi^*_k < \frac{1}{2}$

Table OA3: The Beliefs of the Common Player

For each cell in Table OA3, we record the optimal decision in Table A1(b). □

We restate Lemma A3 before we prove it.

Lemma A3. Equilibrium After the Common Player.

Let $g \in \{A, B\}$.

LA3.1 : (Player $k+1^g$*.) In equilibrium,* $\bar{d}^g_k = (\bar{d}^g_{k-1}, d_k)$ *belongs to a row in Table A2(a) and player* $k+1^g$*'s equilibrium strategy* $\sigma^{g*}_{k+1}(\bar{d}^g_k, x^g_{k+1})$ *is given by the last two columns.*

LA3.2 : (Player $k+2^g$*.) In equilibrium,* $\bar{d}^g_{k+1} = (\bar{d}^g_{k-1}, d_k, d^g_{k+1})$ *belongs to a row in Table A2(b) and player* $k+2^g$*'s equilibrium strategy* $\sigma^{g*}_{k+2}(\bar{d}^g_{k+1}, x^g_{k+2})$ *is given by the last two columns.*

LA3.3 : (Subsequent players.) Let $i > k+2$*. In equilibrium,* $\bar{d}^g_{i-1} = (\bar{d}^g_{k-1}, d_k, d^g_{k+1}, d^g_{k+2}, \ldots, d^g_{i-1})$ *belongs to a row in Table A2(c) and player* i^g*'s equilibrium strategy* $\sigma^{g*}_i(\bar{d}^g_{i-1}, x^g_i)$ *is given by*

the last two columns.

		$\sigma^{g*}_{k+1}(\bar{d}^g_k, x^g_{k+1})$	
$\bar{d}^g_{k-1}$	d_k	H	L
D^a_{k-1}	a	a	a
	r	a	r
D^b_{k-1}	a	a	a
	r	r	r
D^r_{k-1}	a	a	r
	r	r	r

(a) Player $k+1^g$

			$\sigma^{g*}_{k+2}(\bar{d}^g_{k+1}, x^g_{k+2})$	
$\bar{d}^g_{k-1}$	d_k	d^g_{k+1}	H	L
D^a_{k-1}	a	a	a	a
	r	a	a	r
	r	r	r	r
D^b_{k-1}	a	a	a	a
	r	r	r	r
D^r_{k-1}	a	a	a	a
	a	r	a	r
	r	r	r	r

(b) Player $k+2^g$

				$\sigma^{g*}_i(\bar{d}^g_{i-1}, x^g_i)$	
$\bar{d}^g_{k-1}$	d_k	d^g_{k+1}	$(d^g_{k+2}, \ldots, d^g_{i-1})$	H	L
D^a_{k-1}	a	a	$(a, \ldots, a)$	a	a
	r	a	D^a_{i-k-2}	a	a
			D^b_{i-k-2}	a	r
			D^r_{i-k-2}	r	r
	r	r	$(r, \ldots, r)$	r	r
D^b_{k-1}	a	a	$(a, \ldots, a)$	a	a
	r	r	$(r, \ldots, r)$	r	r
D^r_{k-1}	a	a	$(a, \ldots, a)$	a	a
	a	r	D^a_{i-k-2}	a	a
			D^b_{i-k-2}	a	r
			D^r_{i-k-2}	r	r
	r	r	$(r, \ldots, r)$	r	r

(c) Player $i^g > k+2^g$

Table A2: Equilibrium Strategies of Players $k+1$ through N.

We prove each part of Lemma A3 separately. The next lemma gives the probability player $k+1$ observes decision profile $(\bar{d}^g_{k-1}, d_k)$ given the state s.

Lemma OA5. *Let $g \in \{A, B\}$ and $(\bar{d}^g_{k-1}, d_k)$ be such that $\bar{d}^g_{k-1} \in D^a_{k-1} \cup D^b_{k-1} \cup D^r_{k-1}$. Then $P^g_{\sigma^*}(\bar{d}^g_{k-1}, d_k|s)$ is given by Table OA4 for $s \in \{\mathcal{H}, \mathcal{L}\}$.*

	$P^g_{\sigma^*}(\bar{d}^g_{k-1}, d_k\|s)$	
$\bar{d}^g_{k-1}$	$d_k = a$	$d_k = r$
D^a_{k-1}	$P^g_{\sigma^*}(\bar{d}^g_{k-1}\|s)(1 - P(L\|s)^3\Lambda(k-1))$	$P^g_{\sigma^*}(\bar{d}^g_{k-1}\|s)P(L\|s)^3\Lambda(k-1)$
D^b_{k-1}	$P^g_{\sigma^*}(\bar{d}^g_{k-1}\|s)P(H\|s)\lambda^{\frac{k-1}{2}}$ $+P^g_{\sigma^*}(\bar{d}^g_{k-1}\|s)P(H\|s)^2\Lambda(k-1)$	$P^g_{\sigma^*}(\bar{d}^g_{k-1}\|s)P(L\|s)\lambda^{\frac{k-1}{2}}$ $+P^g_{\sigma^*}(\bar{d}^g_{k-1}\|s)P(L\|s)^2\Lambda(k-1)$
D^r_{k-1}	$P^g_{\sigma^*}(\bar{d}^g_{k-1}\|s)P(H\|s)^3\Lambda(k-1)$	$P^g_{\sigma^*}(\bar{d}^g_{k-1}\|s)(1 - P(H\|s)^3\Lambda(k-1))$

Table OA4: Value of $P^g_{\sigma^*}(\bar{d}^g_{k-1}, d_k|s)$

Proof. We illustrate the calculation of the top-right cell of Table OA4, i.e., when $\bar{d}^A_{k-1} \in D^a_{k-1}$ and $d_k = r$. By Lemma A2, the common player chooses r when $\bar{d}^B_{k-1} \in D^r_{k-1}$ and $x_k = L$. Hence,

$$\begin{aligned} P^A_{\sigma^*}(\bar{d}^A_{k-1}, r|s) &= P^A_{\sigma^*}(\bar{d}^A_{k-1}|s)P(L|s)\text{Prob}(D^r_{k-1}|s) \\ &= P^A_{\sigma^*}(\bar{d}^A_{k-1}|s)P(L|s)^3\Lambda(k-1), \end{aligned}$$

where the second line follows from Lemma OA4. The other cells are analogously computed. □

Proof of LA3.1. Without loss we consider player $k+1^A$. Let $(\bar{d}^A_{k-1}, d_k) = (d^A_1, \ldots, d^A_{k-1}, d_k) \in \{D^a_{k-1} \cup D^b_{k-1} \cup D^r_{k-1}\} \times \{a, r\}$. If $\bar{d}^A_{k-1} \in D^a_{k-1}$ or $\bar{d}^A_{k-1} \in D^r_{k-1}$, then there is an odd integer $j < k-1$ such that $d^A_j = d^A_{j+1}$. Let θ denote the smallest such j. (Note that $(d^A_1, \ldots, d^A_\theta)$ is balanced.) If $\bar{d}^A_{k-1} \in D^b_{k-1}$, let $\theta = k-2$.

The belief of player $k+1^A$ is

$$\phi^{A*}_{k+1}(\bar{d}^A_{k-1}, d_k, x^A_{k+1}) = \frac{P(\mathcal{H})P(x^A_{k+1}|\mathcal{H})P^A_{\sigma^*}(\bar{d}^A_{k-1}, d_k|\mathcal{H})}{\sum_{s\in\{\mathcal{H},\mathcal{L}\}} P(s)P(x^A_{k+1}|s)P^A_{\sigma^*}(\bar{d}^A_{k-1}, d_k|s)}.$$

Lemma A1 implies that players 1^A to $\theta + 1^A$ follow their own signals, while players $\theta + 2^A$ to $k - 1^A$ are in a cascade. Thus, $P^A_{\sigma^*}(\bar{d}^A_{k-1}|s) = P(x^A_\theta|s)P(x^A_{\theta+1}|s)(\lambda/2)^{\frac{\theta-1}{2}}$, where $x^A_i = H$ if $d^A_i = a$ and $x^A_i = L$ if $d^A_i = r$ for all $i \in \{\theta, \theta+1\}$. Thus, Lemma OA5 allows us to write the following table, which describes when the belief of player $k+1^A$ is greater than, equal

to, or less than 1/2.

		$\sigma^{A*}_{k+1}(\bar{d}^A_k, x^A_{k+1})$	
$\bar{d}^A_{k-1}$	d_k	H	L
D^a_{k-1}	a	$\phi^{A*}_{k+1} > \frac{1}{2}$	$\phi^{A*}_{k+1} > \frac{1}{2}$
	r	$\phi^{A*}_{k+1} = \frac{1}{2}$	$\phi^{A*}_{k+1} < \frac{1}{2}$
D^b_{k-1}	a	$\phi^{A*}_{k+1} > \frac{1}{2}$	$\phi^{A*}_{k+1} > \frac{1}{2}$
	r	$\phi^{A*}_{k+1} < \frac{1}{2}$	$\phi^{A*}_{k+1} < \frac{1}{2}$
D^r_{k-1}	a	$\phi^{A*}_{k+1} > \frac{1}{2}$	$\phi^{A*}_{k+1} = \frac{1}{2}$
	r	$\phi^{A*}_{k+1} < \frac{1}{2}$	$\phi^{A*}_{k+1} < \frac{1}{2}$

Table OA5: Player $k+1^A$'s Belief

We illustrate the calculation of the top-left cell of Table OA5, i.e., when $\bar{d}^A_{k-1} \in D^a_{k-1}$, $d_k = a$, and $x^A_{k+1} = H$. Then, $P^A_{\sigma^*}(\bar{d}^A_{k-1}, d_k|s) = P^A_{\sigma^*}(\bar{d}^A_{k-1}|s)(1 - P(L|s)^3\Lambda(k-1))$ by Lemma OA5 and $P^A_{\sigma^*}(\bar{d}^A_{k-1}|s) = P(H|s)^2(\lambda/2)^{\frac{\theta-1}{2}}$ since $x^A_\theta = x^A_{\theta+1} = H$. Hence,

$$\phi^{A*}_{k+1}(\bar{d}^A_{k-1}, d_k, H) = \frac{p^3(1-(1-p)^3\Lambda(k-1))}{p^3(1-(1-p)^3\Lambda(k-1)) + (1-p)^3(1-p^3\Lambda(k-1))}.$$

Since $p > 1/2$, then (i) $p^3 > (1-p)^3$ and (ii) $1-(1-p)^3\Lambda(k-1) > 1-p^3\Lambda(k-1)$. Thus, $\phi^{A*}_{k+1}(\bar{d}^A_{k-1}, H) > 1/2$, as shown in Table OA5. For each cell in Table OA5, we record the optimal decision in Table A2(a) when it is unique and have the player follow her signal otherwise. □

Proof of LA3.2. Without loss consider player $k+2^A$. By Lemmas A1 and A2 and by LA3.1, any equilibrium path history $(\bar{d}^A_{k-1}, d_k, d^A_{k+1}) = (d^A_1, \ldots, d^A_{k-1}, d_k, d^A_{k+1})$ belongs to a row in Table A2(b). If $\bar{d}^A_{k-1} \in D^a_{k-1}$ or $\bar{d}^A_{k-1} \in D^r_{k-1}$, then there is an odd integer $j < k-1$ such that $d^A_j = d^A_{j+1}$. Let θ denote the smallest such j. (Note that $(d^A_1, \ldots, d^A_\theta)$ is balanced.) If $\bar{d}^A_{k-1} \in D^b_{k-1}$, let $\theta = k-2$.

The belief of player $k+2^A$ is

$$\phi^{A*}_{k+2}(\bar{d}^A_{k-1}, d_k, d^A_{k+1}, x^A_{k+2}) = \frac{P(\mathcal{H})P(x^A_{k+2}|\mathcal{H})P^A_{\sigma^*}(\bar{d}^A_{k-1}, d_k|\mathcal{H})P^A_{\sigma^*}(d^A_{k+1}|\mathcal{H}, \bar{d}^A_{k-1}, d_k)}{\sum_{s\in\{\mathcal{H},\mathcal{L}\}} P(s)P(x^A_{k+2}|s)P^A_{\sigma^*}(\bar{d}^A_{k-1}, d_k|s)P^A_{\sigma^*}(d^A_{k+1}|s, \bar{d}^A_{k-1}, d_k)}.$$

Lemmas OA5 and LA3.1 allow us to write the following table, which describes when the

belief of player $k+2^A$ is greater than, equal to, or less than 1/2.

			$\phi^{A*}_{k+2}(\cdot)$	
$\bar{d}^A_{k-1}$	d_k	d^A_{k+1}	$x^A_{k+2}=H$	$x^A_{k+2}=L$
D^a_{k-1}	a	a	$\phi^{A*}_{k+2}>\frac{1}{2}$	$\phi^{A*}_{k+2}>\frac{1}{2}$
	r	a	$\phi^{A*}_{k+2}>\frac{1}{2}$	$\phi^{A*}_{k+2}<\frac{1}{2}$
	r	r	$\phi^{A*}_{k+2}<\frac{1}{2}$	$\phi^{A*}_{k+2}<\frac{1}{2}$
D^b_{k-1}	a	a	$\phi^{A*}_{k+2}>\frac{1}{2}$	$\phi^{A*}_{k+2}>\frac{1}{2}$
	r	r	$\phi^{A*}_{k+2}<\frac{1}{2}$	$\phi^{A*}_{k+2}<\frac{1}{2}$
D^r_{k-1}	a	a	$\phi^{A*}_{k+2}>\frac{1}{2}$	$\phi^{A*}_{k+2}>\frac{1}{2}$
	a	r	$\phi^{A*}_{k+2}>\frac{1}{2}$	$\phi^{A*}_{k+2}<\frac{1}{2}$
	r	r	$\phi^{A*}_{k+2}<\frac{1}{2}$	$\phi^{A*}_{k+2}<\frac{1}{2}$

Table OA6: Player $k+2^A$'s Belief

We illustrate the calculation of the left most cell of the second row of Table OA6, i.e., when $\bar{d}^A_{k-1}\in D^a_{k-1}$, $d_k=r$, $d^A_{k+1}=a$, and $x^A_{k+2}=H$. Then, $P^A_{\sigma^*}(\bar{d}^A_{k-1},d_k|s)=P^A_{\sigma^*}(\bar{d}^A_{k-1}|s)P(L|s)^3\Lambda(k-1)$ by Lemma OA5 and $P^A_{\sigma^*}(\bar{d}^A_{k-1}|s)=P(H|s)^2(\lambda/2)^{\frac{\theta-1}{2}}$ since $x^A_\theta=x^A_{\theta+1}=H$. Hence,

$$P^A_{\sigma^*}(\bar{d}^A_{k-1},d_k|s)=P(H|s)^2(\lambda/2)^{\frac{\theta-1}{2}}P(L|s)^3\Lambda(k-1). \tag{1}$$

Since $k+1^A$ follows her signal by LA3.1, $P^A_{\sigma^*}(d^A_{k+1}|s,\bar{d}^A_{k-1},d_k)=P(H|s)$. Thus,

$$\phi^{A*}_{k+2}(\bar{d}^A_{k+1},r,a,H)=\frac{p^4(1-p)^3\Lambda(k-1)}{p^4(1-p)^3\Lambda(k-1)+p^3(1-p)^4\Lambda(k-1)}=p>1/2.$$

For each cell in Table OA6, we record the optimal decision in Table A2(b) when it is unique and have the player follow her signal otherwise. □

Proof of LA3.3. Without loss, consider group A. If player $k+2^A$ is in a cascade, then it is easily seen that every subsequent player is also in a cascade on the same decision. Thus, we focus on the cases where $k+2^A$ follows her signal, i.e., where (i) $\bar{d}^A_{k-1}\in D^a_{k-1}$, $d_k=r$, and $d^A_{k+1}=a$ or (ii) $\bar{d}^A_{k-1}\in D^r_{k-1}$, $d_k=a$, and $d^A_{k+1}=r$. In these cases, the behavior of player

$i^A > k+2$ is given by the following table, which is analogous to Lemma A1.

	$\sigma_i^{A*}(\cdot, d_{k+2}^A, \ldots, d_{i-1}^A, x_i^A)$	
$(d_{k+2}^A, \ldots, d_{i-1}^A)$	H	L
D_{i-k-2}^a	a	a
D_{i-k-2}^b	a	r
D_{i-k-2}^r	r	r

Table OA7: Equilibrium Behavior of Player $i^A > k+2$

Table OA7 gives that player i^A's equilibrium strategy depends only on the decisions of players $k+2^A$ through $i-1^A$ and her own signal.

For decision profiles satisfying (i) or (ii), we show that

$$P_{\sigma^*}^A(\bar{d}_{k-1}^A, d_k, d_{k+1}^A|\mathcal{H}) = P_{\sigma^*}^A(\bar{d}_{k-1}^A, d_k, d_{k+1}^A|\mathcal{L}). \tag{2}$$

To see this, suppose the profile satisfies (i). We have that

$$P_{\sigma^*}^A(\bar{d}_{k-1}^A, d_k, d_{k+1}^A|s) = P_{\sigma^*}^A(\bar{d}_{k-1}^A, d_k|s)P(H|s) = P(H|s)^3(\lambda/2)^{\frac{\theta-1}{2}}P(L|s)^3\Lambda(k-1),$$

where the second equality follows from equation (1). Since the right-hand-side is the same all states s, equation (2) holds. The argument for decision profiles satisfying (ii) is analogous.

Consider player $k+3^A$. Since the decision profile satisfies (i) or (ii), we have

$$\begin{aligned}
&\phi_{k+3}^{A*}(\bar{d}_{k-1}^A, d_k, d_{k+1}^A, d_{k+2}^A, x_{k+3}^A) \\
&= \frac{P(\mathcal{H})P(x_{k+3}^A|\mathcal{H})P_{\sigma^*}^A(\bar{d}_{k-1}^A, d_k, d_{k+1}^A|\mathcal{H})P_{\sigma^*}^A(d_{k+2}^A|\mathcal{H}, \bar{d}_{k-1}^A, d_k, d_{k+1}^A)}{\sum_{s\in\{\mathcal{H},\mathcal{L}\}} P(s)P(x_{k+3}^A|s)P_{\sigma^*}^A(\bar{d}_{k-1}^A, d_k, d_{k+1}^A|s)P_{\sigma^*}^A(d_{k+2}^A|s, \bar{d}_{k-1}^A, d_k, d_{k+1}^A)} \\
&= \frac{P(\mathcal{H})P(x_{k+3}^A|\mathcal{H})P_{\sigma^*}^A(d_{k+2}^A|\mathcal{H}, \bar{d}_{k-1}^A, d_k, d_{k+1}^A)}{\sum_{s\in\{\mathcal{H},\mathcal{L}\}} P(s)P(x_{k+3}^A|s)P_{\sigma^*}^A(d_{k+2}^A|s, \bar{d}_{k-1}^A, d_k, d_{k+1}^A)} \\
&= \frac{P(\mathcal{H})P(x_{k+3}^A|\mathcal{H})P(x_{k+2}^A|\mathcal{H})}{\sum_{s\in\{\mathcal{H},\mathcal{L}\}} P(s)P(x_{k+3}^A|s)P(x_{k+2}^A|s)} \text{ with } x_{k+2}^A = \begin{cases} H & \text{if } d_{k+2}^A = a \\ L & \text{if } d_{k+2}^A = r \end{cases},
\end{aligned}$$

where the second equality follows from equation (2) and the third equality follows from the fact that $k+2^A$ follows her signal (see LA3.2). It is easily seen that $\phi_{k+3}^{A*}(\cdot, H) \geq 1/2$ and $\phi_{k+3}^{A*}(\cdot, L) \leq 1/2$ for all d_{k+2}^A, so Table OA7 describes the equilibrium decision of player $k+3^A$.

Suppose Table OA7 describes the equilibrium behavior of players $k+3^A$ to $i-1^A$. We show that the table also describes the behavior of player i^A. Write

$$P_{\sigma^*}^A(\bar{d}_{k-1}^A, d_k, d_{k+1}^A, \ldots, d_{i-1}^A|s) = P_{\sigma^*}^A(\bar{d}_{k-1}^A, d_k, d_{k+1}^A|s)P_{\sigma^*}^A(d_{k+2}^A, \ldots, d_{i-1}^A|s, \bar{d}_{k-1}^A, d_k, d_{k+1}^A).$$

Since the decision profile satisfies (i) or (ii), there are three cases by the induction hypothesis: (a) $(d^A_{k+2}, \ldots, d^A_{i-1}) \in D^b_{i-k-2}$, (b) $(d^A_{k+2}, \ldots, d^A_{i-1}) \in D^a_{i-k-2}$, and (c) $(d^A_{k+2}, \ldots, d^A_{i-1}) \in D^r_{i-k-2}$.

Case (a): If $(d^A_{k+2}, \ldots, d^A_{i-1}) \in D^b_{i-k-2}$, then the induction hypothesis gives

$$P^A_{\sigma^*}(d^A_{k+2}, \ldots, d^A_{i-1}|s, \bar{d}^A_{k-1}, d_k, d^A_{k+1}) = \prod_{j=k+2}^{i-1} P(x^A_j|s) \text{ where } x^A_j = \begin{cases} H & \text{if } d^A_j = a \\ L & \text{if } d^A_j = r. \end{cases}$$

Thus i's belief given signal x^A_i is

$$\begin{aligned}
&\phi^{A*}_i(\bar{d}^A_{k-1}, d_k, d^A_{k+1}, d^A_{k+2}, \ldots, d^A_{i-1}, x^A_i) \\
&= \frac{P(\mathcal{H})P(x^A_i|\mathcal{H})P^A_{\sigma^*}(\bar{d}^A_{k-1}, d_k, d^A_{k+1}|\mathcal{H})P^A_{\sigma^*}(d^A_{k+2}, \ldots, d^A_{i-1}|\mathcal{H}, \bar{d}^A_{k-1}, d_k, d^A_{k+1})}{\sum_{s\in\{\mathcal{H},\mathcal{L}\}} P(s)P(x^A_i|s)P^A_{\sigma^*}(\bar{d}^A_{k-1}, d_k, d^A_{k+1}|s)P^A_{\sigma^*}(d^A_{k+2}, \ldots, d^A_{i-1}|s, \bar{d}^A_{k-1}, d_k, d^A_{k+1})} \\
&= \frac{P(\mathcal{H})P(x^A_i|\mathcal{H})P^A_{\sigma^*}(d^A_{k+2}, \ldots, d^A_{i-1}|\mathcal{H}, \bar{d}^A_{k-1}, d_k, d^A_{k+1})}{\sum_{s\in\{\mathcal{H},\mathcal{L}\}} P(s)P(x^A_i|s)P^A_{\sigma^*}(d^A_{k+2}, \ldots, d^A_{i-1}|s, \bar{d}^A_{k-1}, d_k, d^A_{k+1})} \\
&= \begin{cases} \frac{P(\mathcal{H})P(x^A_i|\mathcal{H})}{\sum_{s\in\{\mathcal{H},\mathcal{L}\}} P(s)P(x^A_i|s)} & \text{if } i \text{ is odd} \\ \frac{P(\mathcal{H})P(x^A_i|\mathcal{H})P(x^A_{i-1}|\mathcal{H})}{\sum_{s\in\{\mathcal{H},\mathcal{L}\}} P(s)P(x^A_i|s)P(x^A_{i-1}|s)} & \text{if } i \text{ is even.} \end{cases}
\end{aligned}$$

We have $\phi^{A*}_i(\cdot, H) \geq 1/2$ for all $\bar{d}^A_{i-1}$ and $\phi^*_i(\cdot, L) \leq 1/2$ for all $\bar{d}^A_{i-1}$, so player i^A follows her own signal as in Table OA7.

Case (b): If $(d^A_{k+2}, \ldots, d^A_{i-1}) \in D^a_{i-k-2}$, then there is an odd integer j between $k+2$ and $i-1$ such that $d^A_j = d^A_{j+1} = a$. The induction hypothesis gives

$$P^A_{\sigma^*}(d^A_{k+2}, \ldots, d^A_{i-1}|s, \bar{d}^A_{k-1}, d_k, d^A_{k+1}) = (\lambda/2)^{\frac{\theta-k-2}{2}} P(H|s)^2.$$

Hence

$$\phi^{A*}_i(\bar{d}^A_{k-1}, d_k, d^A_{k+1}, d^A_{k+2}, \ldots, d^A_{i-1}, x^A_i) = \frac{P(\mathcal{H})P(x^A_i|\mathcal{H})P(H|\mathcal{H})^2}{\sum_{s\in\{\mathcal{H},\mathcal{L}\}} P(s)P(x^A_i|s)P(H|s)^2}.$$

It is readily verified that $\phi^{A*}_i(\cdot, x_i) > 1/2$ for $x_i \in \{H, L\}$. Thus, $\sigma^*_i(\cdot, x_i) = a$ as in Table OA7. The proof for case (c) is analogous and is omitted.

We now construct Table A2(c). If $(\bar{d}^A_{k-1}, d_k, d^A_{k+1})$ is such that player $k+2^A$ is in a cascade, then player i^A is in the same cascade. Thus, we obtain rows 1, 5, 6, 7, 8, and 12 of Table A2(c). If $(\bar{d}^A_{k-1}, d_k, d^A_{k+1})$ is such that $k+2^A$ follows her signal, then Table OA7 pins down the equilibrium decisions of all subsequent players. Thus, we obtain rows 2, 3, 4, 9, 10, and 11 of Table A2(c). □

Proof of Proposition 4. The proof is computational. We first consider the case of $i < k$, then we consider the case of $i \geq k$. Without loss, we take i to be in group A.

Case (i), $i < k$: Since i is less than k, Lemma A1 gives that i chooses a if (i) $\bar{d}^A_{i-1} \in D^b_{i-1}$ and $x^A_i = H$ or (ii) $\bar{d}^A_{i-1} \in D^A_{i-1}$. Thus, the probability i chooses a given the state is s is

$$\text{Prob}(i \text{ chooses } a|s) = \begin{cases} \lambda^{\frac{i-2}{2}} P(H|s) + \frac{(P(H|s))^2}{1-\lambda}(1-\lambda^{\frac{i-2}{2}}) & \text{for } i \text{ even} \\ \lambda^{\frac{i-1}{2}} P(H|s) + \frac{(P(H|s))^2}{1-\lambda}(1-\lambda^{\frac{i-1}{2}}) & \text{for } i \text{ odd.} \end{cases}$$

It follows that i's ex-ante payoff is

$$w(i,k) = \text{Prob}(i \text{ chooses } a|\mathcal{H})P(\mathcal{H}) - \text{Prob}(i \text{ chooses } a|\mathcal{L})P(\mathcal{L}),$$

which simplifies to the stated expression. We ignore the events where i chooses r because choosing r always gives zero.

Case (ii), $i \geq k$: Let $g \in \{A, B\}$ and let $c^g \in \{a, n, r\}$ denote the type of cascade in group g before k – i.e., $c^g = a \iff \bar{d}^g_{k-1} \in D^a_{k-1}$, $c^g = n \iff \bar{d}^g_{k-1} \in D^b_{k-1}$, and $c^g = r \iff \bar{d}^g_{k-1} \in D^r_{k-1}$. Let $C_2 = \{(a,r),(r,a)\}$ be the set of cascades on different decisions and $C_1 = \{a,n,r\}^2 \backslash C_2$ be the set of all other cascades.

We have that $w(i,k) = g_1(i,k) + g_2(i,k)$, with

$$g_1(i,k) = \sum_{\substack{s \in \{\mathcal{H},\mathcal{L}\} \\ (c_A,c_B) \in C_1}} u(a,s)\text{Prob}(i \text{ chooses } a|s,c_A,c_B)\text{Prob}(c^A|s)\text{Prob}(c^B|s)P(s)$$

$$g_2(i,k) = \sum_{\substack{s \in \{\mathcal{H},\mathcal{L}\} \\ (c_A,c_B) \in C_2}} u(a,s)\text{Prob}(i \text{ chooses } a|s,c_A,c_B)\text{Prob}(c^A|s)\text{Prob}(c^B|s)P(s).$$

Here, $\text{Prob}(i \text{ chooses } a|s,c^A,c^B)$ is the probability that i chooses a in state s, and $\text{Prob}(c|s)$ is given by Lemma OA4.

First, we calculate g_1. Proposition 3 gives that if $(c^A,c^B) \in \{(a,a),(a,n),(n,a)\}$, then i chooses a with certainty, so $\text{Prob}(i \text{ chooses } a|s,c^A,c^B) = 1$ for both states $\mathcal{H}$ and $\mathcal{L}$. If $(c^A,c^B) = (n,n)$, then all players after k choose d_k with certainty. Since Lemma A2 gives $d_k = a$ if and only if $x_k = H$, we have $d_k = a$ with probability $P(H|s)$. Hence, $\text{Prob}(i \text{ chooses } a|\mathcal{H},c^A,c^B) = p$ and $\text{Prob}(i \text{ chooses } a|L,c^A,c^B) = 1-p$. Plugging these conditional probabilities into the expression for g_1 and simplifying gives

$$g_1(i,k) = \frac{(2p-1)(1-\lambda^k)}{2(1-\lambda)} = \frac{2p-1}{2}\Lambda(2k).$$

Now we calculate g_2. Since the cascades are in C_2, we successively employ Lemma A3 to calculate the probability that i chooses a. Table OA8 gives the results of these computations.

	Prob(i chooses $a \mid s, c^A, c^B$)	
Player Position	$(c_A, c_B) = (a, r)$	$(c_A, c_B) = (r, a)$
k	$P(H\|s)$	$P(H\|s)$
$k+1$	$P(H\|s) + P(L\|s)P(H\|s)$	$P(H\|s)^2$
$k+2$	$P(H\|s) + P(L\|s)P(H\|s)^2$	$P(H\|s)^2 + P(L\|s)P(H\|s)^2$
$i > k+2$ and i odd	$P(H\|s)$ $+P(L\|s)P(H\|s)^2\lambda^{\frac{i-k-2}{2}}$ $+P(L\|s)P(H\|s)^3\Lambda(i-k-2)$	$P(H\|s)^2$ $+P(L\|s)P(H\|s)^2\lambda^{\frac{i-k-2}{2}}$ $+P(L\|s)P(H\|s)^3\Lambda(i-k-2)$
$i > k+2$ and i even	$P(H\|s)$ $+P(L\|s)P(H\|s)^2\lambda^{\frac{i-k-3}{2}}$ $+P(L\|s)P(H\|s)^3\Lambda(i-k-3)$	$P(H\|s)^2$ $+P(L\|s)P(H\|s)^2\lambda^{\frac{i-k-3}{2}}$ $+P(L\|s)P(H\|s)^3\Lambda(i-k-3)$

Table OA8: Probability Player i Chooses a

We illustrate the construction of the bottom-left cell: $(c^A, c^B) = (a, r)$ for $i > k+2$ and i even. We begin by noting that successive application of Lemma A3 shows that the only histories i observes in equilibrium are those listed in Table A2(c). Thus, i chooses a if (i) $d_k = a$ (i.e., a cascade on a immediately after k), (ii) $d_k = r$, $d^A_{k+1} = a$, $(d^A_{k+2}, \ldots, d^A_{i-1}) \in D^b_{i-k-2}$ (i.e., no cascade after k), and $x^A_i = H$, or (iii) $d_k = r$, $d^A_{k+1} = a$, and $(d^A_{k+2}, \ldots, d^A_{i-1}) \in D^a_{i-k-2}$ (i.e., a cascade on a at some point after k). Since Lemma A2 gives $d_k = a \iff x_k = H$, the (conditional) probability of (i) is $P(H|s)$. As to the probability of (ii): Lemmas A2 and A3 give that $x_k = L$, $x^A_{k+1} = H$, and $x_i = H$. In addition, Lemma A3 gives that players $k+2^A$ to $i-1^A$ follow their signals, which implies that the probability of $(d^A_{k+2}, \ldots, d^A_{i-1}) \in D^b_{i-k-2}$ is $\lambda^{\frac{i-k-3}{2}}$. Thus, the probability of (ii) is $P(L|s)P(H|s)^2\lambda^{\frac{i-k-3}{2}}$. As to the probability of (iii): We have $x_k = L$ and $x^A_{k+1} = H$. Since $(d^A_{k+2}, \ldots, d^A_{i-1}) \in D^a_{i-k-2}$, there is an odd $\theta < i-1$ such that $(d^A_{k+2}, \ldots, d^A_\theta)$ is balanced and $d^A_\theta = d^A_{\theta+1} = \cdots = d^A_{i-1} = a$. By Lemma A3, players $k+2^A$ to $\theta+1^A$ follow their own signal, while players θ^A+2 to $i-1^A$ are in a cascade on a. This implies the probability $(d^A_{k+2}, \ldots, d^A_{i-1}) \in D^a_{i-k-2}$ is $P(H|s)^2\Lambda(i-k-3)$ (because there are $\frac{i-k-3}{2}$ places to put θ). Thus, the probability of (iii) is $P(L|s)P(H|s)^3\Lambda(i-k-3)$. Summing the probabilities of (i), (ii), and (iii) gives the cell.

Plugging the values from Table OA8 into the expression for g_2 and simplifying gives

$$g_2(i,k) = \begin{cases} (2p-1)(\frac{\lambda}{2}\Lambda(k-1))^2 & \text{if } i = k \text{ or } i = k+1 \\ (2p-1)(\frac{\lambda}{2}\Lambda(k-1))^2(1+\frac{\lambda}{2}\Lambda(i-k)) & \text{if } i \geq k+2. \end{cases}$$

The display equation of the proposition follows. □

4 Examples

In this section, we first give an Illustrative Example that illustrates the results in the main text and provides intuition. Subsequently, we introduce two additional examples that explore extensions of an interacting cascade and are motivated by the fact that groups may have multiple points of contact. The first of these examples has two common players, and the second has an "anonymous" common player, i.e., one whose position is unknown to the other players. In both examples, cascades spill over, break, and are triggered in similar ways as in an interacting cascade, indicating that our results are qualitatively robust. Nevertheless, there are differences – e.g., when (i) there are two common players, both groups' cascades can be broken; and when (ii) the common player is anonymous, the common player cannot trigger a cascade when she moves late.

The Illustrative Example

To illustrate our model and give intuition for our core results, we consider an example wherein a group of co-workers, A(lice), B(ill), D(avid), E(mma), M(ichael), S(ue), and W(alter), each decide whether to watch Netflix's latest new show on the night it is released. The show may either be good (state $\mathcal{H}$) or bad (state $\mathcal{L}$) with equal probability. The players would like to watch a good show, i.e., choose a if the state is $\mathcal{H}$, and avoid watching a bad show, i.e., choose r if the state is $\mathcal{L}$.

The problem is that no one knows the quality of the show before it is released. However, everyone has seen a promo about it (e.g., a billboard, online clip, etc.), i.e., they have each received an informative private signal. We assume that the promo correctly indicates the quality of the show with two-thirds probability, i.e., $p = 2/3$. In addition, some of the coworkers subscribe to each other's Netflix "Social" feeds, and can observe whether their

peers are also watching the show.[2] Specifically, Bill, David, and Walter, collectively the Men, all subscribe to each other's feeds. Likewise, Alice, Emma, and Sue, collectively the Women, all subscribe to each other's feeds. Michael subscribes to everyone's feed.

Once Netflix releases the show, the co-workers make their viewing decisions in alphabetical order. When a player moves, (s)he observes the feeds (s)he is subscribed to and her signal, and then makes her decision. To round out the example, we suppose that watching a good show pays 1, while not watching pays 0, and watching a bad show pays -1. The arrangement of players is illustrated in Figure OA1.

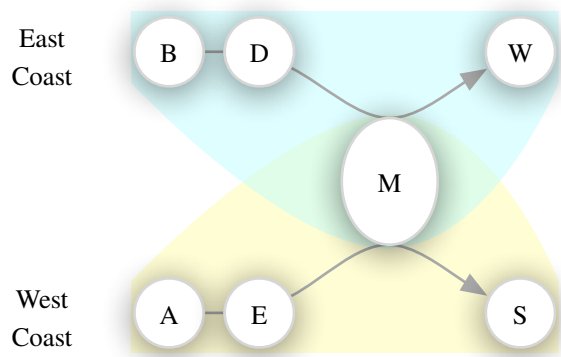

Figure OA1: Arrangement of Players in Illustrative Example

Basic Cascades. To see how a cascade forms, we focus on the Men and suppose, for the moment, that the Women do not exist. Suppose that Bill's (private) signal is L, i.e., the promo indicates to him that Netflix's show is likely bad. Then, his belief is

$$\phi_B^*(L) = \frac{P(\mathcal{H})P(L|\mathcal{H})}{\sum_{s\in\{\mathcal{H},\mathcal{L}\}} P(s)P(L|s)} = \frac{\frac{1}{2}(1-p)}{\frac{1}{2}(1-p)+\frac{1}{2}p} = \frac{1}{3},$$

and he optimally chooses not to watch, i.e., $\sigma_B^*(L) = r$. If B's signal had been H, then $\phi_B^*(H) = 2/3$ and B would have chosen a. B's decision thus reveals his signal to David, Michael, and Walter. Note that in state $\mathcal{H}$ the probability B chooses r is $P_{\sigma^*}(r|\mathcal{H}) = 1-p = 1/3$, while in state $\mathcal{L}$ we have $P_{\sigma^*}(r|\mathcal{L}) = p = 2/3$.

[2] Netflix offers its subscribers the ability to automatically share what they watch to social media sites (e.g., Facebook), via a Netflix Social feed. See www.netflix.com for details. While this feed also shows one's ratings, we have in mind an environment where players are making their decisions so quickly that they have yet to rate the show.

David sees Bill choose r and infers that B received a signal of L. If D's signal is also L (i.e., $\bar{d}_B = r$ and $x_D = L$ in the notation of the model) then his belief is

$$\phi_D^*(r, L) = \frac{P(\mathcal{H})P(L|\mathcal{H})P_{\sigma^*}(r|\mathcal{H})}{\sum_{s\in\{\mathcal{H},\mathcal{L}\}} P(s)P(L|s)P_{\sigma^*}(r|s)} = \frac{\frac{1}{2}(1-p)^2}{\frac{1}{2}(1-p)^2 + \frac{1}{2}p^2} = \frac{1}{5}.$$

Thus, D also chooses r. If D's signal had been H, then $\phi_D^*(r, H) = 1/2$ and D would have chosen a, following his signal. D's decision therefore also reveals his signal.

Michael, after observing Bill and David choose not to watch, i.e., $\bar{d}_D = (r, r)$, infers that they both received signals of L. Thus,

$$\phi_M^*(r, r, x_M) = \begin{cases} 3/7 & \text{if } x_M = H \\ 1/9 & \text{if } x_M = L, \end{cases}$$

which is less than 1/2 for both realizations of x_M. Hence, M optimally follows the decisions of B and D and also chooses r. Consequently, $\bar{d}_M = (r, r, r)$. As M's decision does not depend on his signal, his decision is uninformative to Walter. Therefore, W has the same belief as M, i.e., $\phi_W^*(\bar{d}_M, x) = \phi_M^*(\bar{d}_D, x)$ for $x \in \{L, H\}$, and also chooses r.

When Bill and David both choose r, then Michael and Walter are in a cascade. M chooses r because the information revealed by B and D's decisions is strong enough to "outweigh" his signal. Since M's decision is uninformative, then W is in the same situation as M and makes the same decision. This leads to our first observation.

Observation OA1: *In a basic cascade, if the private signals of the first two players (e.g., Bill and David) are the same, then a cascade emerges, i.e., the third player and all subsequent players make the same decision as the first two players.*

If the signals of the first two players are not the same, then Michael is not in a cascade. If David's signal had been H rather than L, he would have chosen a. In this case, M infers $(x_B, x_D) = (L, H)$ and has a belief

$$\phi_M^*(r, a, x_M) = \begin{cases} 2/3 & \text{if } x_M = H \\ 1/3 & \text{if } x_M = L. \end{cases}$$

Thus, M's decision would depend on his signal.

Interacting Cascades. To see how cascades interact, we return to our initial setting with both Men and Women. In light of Observation OA1, we say (in the context of this example)

that group g is in a cascade if the decisions of the first two players in g are the same. We illustrate equilibrium play for the common player and the subsequent players when: (i) one group is in a cascade and the other is not, and (ii) both groups are in cascades on different decisions.

To examine (i), suppose that the Men are in a cascade on not watching, i.e., both Bill and David choose r, and that the Women are not in a cascade, e.g., Alice chooses a and Emma chooses r. Thus, $\bar{d}_D^M = (r, r)$ and $\bar{d}_E^W = (a, r)$. Since these players follow their private signals, Michael infers that $(x_B, x_D) = (L, L)$ and $(x_A, x_E) = (H, L)$. M's belief given his signal x_M is

$$\phi_M^*(\bar{d}_D^M, \bar{d}_E^W, x_M) = \begin{cases} \frac{\frac{1}{2}p^2(1-p)^3}{\frac{1}{2}p^2(1-p)^3+\frac{1}{2}p^3(1-p)^3} = \frac{1}{3} & \text{if } x_M = H \\ \frac{\frac{1}{2}p(1-p)^4}{\frac{1}{2}p(1-p)^4+\frac{1}{2}p^4(1-p)} = \frac{1}{9} & \text{if } x_M = L. \end{cases}$$

Thus, M decides r, regardless of his signal, following the decision of the group that is in a cascade.

Table OA9 summarizes how Michael's (equilibrium) decision depends on his signal and the private signals of Bill, David, Alice, and Emma. M will choose r if at least three of these five signals are L and will choose a otherwise.

	$x_M = H$				$x_M = L$			
Men (x_B, x_D)	Women (x_A, x_E)				Women (x_A, x_E)			
	HH	HL	LH	LL	HH	HL	LH	LL
HH	a	a	a	a	a	a	a	r
HL	a	a	a	r	a	r	r	r
LH	a	a	a	r	a	r	r	r
LL	a	r	r	r	r	r	r	r

Table OA9: Michael's Decisions

Continuing the example, suppose that Michael indeed chooses r. Walter observes $\bar{d}_M^M = (r, r, r)$, from which he infers that $x_B = x_D = L$. He cannot infer (x_A, x_E, x_M). However, W knows that the only signal profiles, $(x_B, x_D, x_A, x_E, x_M)$, which are consistent with $\bar{d}_M^M$ are those in the last row of Table OA9 *except* for the left-most cell (L, L, H, H, H). W calculates

that the probability of $\bar{d}_M^M$ in state $\mathcal{H}$ is

$$P_{\sigma^*}^M(\bar{d}_M^M|\mathcal{H}) = 3p^2(1-p)^3 + 3p(1-p)^4 + (1-p)^5 = \frac{19}{243},$$

and in state $\mathcal{L}$ is

$$P_{\sigma^*}^M(\bar{d}_M^M|\mathcal{L}) = 3p^3(1-p)^2 + 3p^4(1-p) + p^5 = \frac{104}{243}.$$

Thus, W's belief, given signal x_W, is[3]

$$\phi_W^*(\bar{d}_M^M, x_W) = \begin{cases} \frac{19}{71} & \text{if } x_W = H \\ \frac{19}{227} & \text{if } x_W = L. \end{cases} \tag{3}$$

Thus, W chooses r, regardless of his signal. This leads to our second observation.

Observation OA2a: *If group g is in a cascade and the other group is not, then the decision of the common player follows the cascade and the cascade in g continues.*

Let us consider the Women. Sue observes $\bar{d}_M^W = (a, r, r)$ and infers $(x_A, x_E) = (H, L)$. Since Michael decided r, S knows that there were at least three L signals among $(x_B, x_D, x_A, x_E, x_M)$. Since $x_A = H$ and $x_E = L$, then S knows that Bill, David, and Michael collectively received at least two L signals. The probability of this is $3p(1-p)^2 + (1-p)^3$ if the state is $\mathcal{H}$.[4] S calculates that the probability of $\bar{d}_M^W$ in state $\mathcal{H}$ is

$$P_{\sigma^*}^W(\bar{d}_M^W|\mathcal{H}) = p(1-p)[3p(1-p)^2 + (1-p)^3] = \frac{14}{243}$$

and, in an analogous manner, that the probability of $\bar{d}_M^W$ in state $\mathcal{L}$ is

$$P_{\sigma^*}^W(\bar{d}_M^W|\mathcal{L}) = p(1-p)[3p^2(1-p) + p^3] = \frac{40}{243}.$$

S's belief, given signal x_S, is

$$\phi_S^*(\bar{d}_M^W, x_S) = \begin{cases} \frac{7}{17} & \text{if } x_S = H \\ \frac{7}{47} & \text{if } x_S = L. \end{cases}$$

[3]For instance,

$$\phi_W^\star(\bar{d}_M^M, H) = \frac{\frac{1}{2}pP_{\sigma^*}^M(\bar{d}_M^M|\mathcal{H})}{\frac{1}{2}pP_{\sigma^*}^M(\bar{d}_M^M|\mathcal{H}) + \frac{1}{2}(1-p)P_{\sigma^*}^M(\bar{d}_M^M|\mathcal{L})} = \frac{19}{71}.$$

[4]There are three ways that two of the players could receive an L signal, and one way all three players could receive an L signal.

Thus, S chooses r regardless of her signal. Intuitively, M's decision conveys enough information about the state to outweigh S's private signal and the information revealed by Alice and Emma's decisions. The cascade on r "spills over" from the Men to the Women. This leads to our third observation.

Observation OA2b: *If group g is in a cascade and the other group $-g$ is not, then subsequent players in $-g$ follow the cascade. That is, the cascade spills over from one group to the other.*

We now consider the case in which both groups are in cascades on different decisions. Suppose that the Men are in a cascade on not watching, i.e., both Bill and David choose r, and that the Women are in a cascade on watching, i.e., both Alice and Emma choose a. Then Michael infers $(x_B, x_D) = (L, L)$ and $(x_A, x_E) = (H, H)$. Table OA9 shows that M resolves his decision in favor of his signal.

Suppose that $x_M = L$ and thus $d_M = r$. Then $\bar{d}_M^M = (r, r, r)$ and Walter's belief is given by (3), so he optimally chooses r and the cascade in the Men's group continues. Sue, however, observes $\bar{d}_M^W = (a, a, r)$ and infers $(x_A, x_E) = (H, H)$. Table OA9 shows that when $x_A = x_E = H$, then Michael only chooses r if $(x_B, x_D, x_M) = (L, L, L)$. Hence, S's history effectively contains three L signals and two H signals. S's belief, given signal x_S, is

$$\phi_S^*(\bar{d}_M^W, x_S) = \begin{cases} \frac{1}{2} & \text{if } x_S = H \\ \frac{1}{5} & \text{if } x_S = L. \end{cases}$$

Hence, S chooses a if $x_S = H$ and r if $x_S = L$, and so S is not in a cascade. Intuitively, M's decision to not watch conveys enough information to S for her to doubt her group's cascade on watching, but not enough to give rise to a new cascade. This leads to our next observation.

Observation OA3: *If both groups are in cascades on different decisions, then the cascade that agrees with the common player's decision continues, while the cascade that disagrees with the common player's decision ends.*

Observations OA2a, OA2b, and OA3 allow us to write Table OA10, a description of how cascades interact when the Men are in a cascade on a. In the table, "a" denotes a cascade on a, "r" a cascade on r, and "n/a" denotes no cascade. For instance, the second row of the

table shows that if the Men are in a cascade on a (before Michael moves) and the Women are not in a cascade, then M chooses a, the Men remain in a cascade on a, and this cascade is transmitted to the Women.

Existing Cascade			Subsequent Cascade	
Men	Women	d_M	Men	Women
	a	a	a	a
a	n/a	a	a	a
	r	a	a	n/a
	r	r	n/a	r

Table OA10: Michael's Effect on Cascades in Both Groups

Table OA10 shows our two findings for this example: (i) if one group is in a cascade and the other is not, then the cascade is transmitted through the common player to the second group, and (ii) when two cascades on opposite decisions meet, one ends. It also shows that when both groups cascade on the same decision, then interaction has no effect.

Welfare. When is interaction via a common player welfare improving? Table OA10 indicates that the common player Michael sometimes generates a benefit. For instance, if the state is $\mathcal{H}$, the Men are in cascade on a (the correct decision), and the Women are in a cascade on r (the incorrect decision), then when M chooses a he ends the Women's incorrect cascade and raises Sue's payoff. This comes with the risk, however, that M may choose r and end the Men's correct cascade, thereby decreasing Walter's payoff. To get a handle on this we need to compare the (ex-ante) equilibrium payoffs of Walter and Sue with and without a common player.

In a basic cascade, where Michael is a regular player, Walter's (and Sue's) equilibrium payoff is $w^B = 13/54$. In an interacting cascade, where M is a common player, W's equilibrium payoff is $w^I = 47/162$. Since $w^I - w^B = 4/81 > 0$, W and S are better off when groups interact than when they do not.[5]

The presence of a common player raises payoffs by increasing the chance that players who follow him choose the correct decision. He accomplishes this by (i) ending incorrect cascades more often than correct cascades and (ii) by transmitting correct cascades (to a

[5] Equilibrium payoffs in basic and interacting cascades are given in Proposition 5.

group not in a cascade) more often than incorrect cascades. For this illustrative example, we can show that interaction always raises welfare. In general, we have the following result from Proposition 5.

Observation OA4: *Ex-ante (equilibrium) payoffs in the interacting cascade are (strictly) higher than ex-ante payoffs in the basic cascade if p is sufficiently high or k is sufficiently large.*

Two Common Players

In this subsection, we consider a simple example with two common players and characterize its equilibrium.

Setup. Two groups of six players each share common players in the third and fifth positions. The arrangement of players is illustrated in Figure OA2. We assume $p = 2/3$. All other features of the environment are as in the main text.

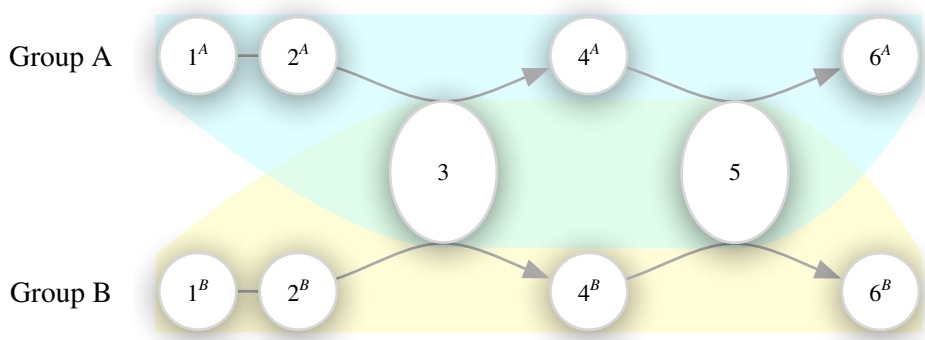

Figure OA2: Multiple Common Players

Equilibrium. It is easily seen that player 1^g to 4^g behave as in the main text. Thus, we study players 5 and 6^g. Since player 5 observes the entire history of play, it is straightforward to compute her equilibrium path play.

Lemma OA6. Equilibrium Play of the Second Common Player.
If (i) both groups are in cascades on the same decision (prior to the first common player) or (ii) one group is in a cascade and the other group is not in a cascade, then player 5 follows the cascade. If both groups are in opposing cascades, then player 5 follows players 3, 4^A, and 4^B if their decisions agree. In all other cases, 5 follows her own signal.

The inference problem facing players 6^A and 6^B is complex. Fortunately, the decisions of the common players usually reveal the decisions of all the other players. For instance, observing the history $(d_1^A, d_2^A, d_3, d_4^A, d_5) = (a, a, a, a, r)$ reveals to 6^A that $(d_1^B, d_2^B, d_4^B) = (r, r, r)$. (Simply, Lemma OA6 implies that 5 chooses r after observing $(d_1^A, d_2^A, d_3, d_4^A) = (a, a, a, a)$ only if $(d_1^B, d_2^B) = (r, r)$ and $d_4^B = r$.) Thus, 6^A infers that the history contains three H signals and four L signals, for a surplus of one L signal. Hence, after her own signal of H (L), Bayes' rule gives that 6^A's belief equals 1/2 (is less than 1/2). Analogous reasoning establishes the following.

Lemma OA7. Equilibrium Play of Player 6^g.
(i) If group g is not in a cascade (prior to player 3), then (i.a) 6^g is in a cascade on player 3's decision when players 3 and 5 make the same decision and (i.b) follows her own signal otherwise.
(ii) If group g is in a cascade and 3's decision agrees with the cascade, then (ii.a) 6^g is in the same cascade when 5's decision agrees with the cascade and (ii.b) 6^g follows her signal otherwise.
(iii) If group g is in a cascade and 3's decision disagrees with the cascade, then (iii.a) 6^g is in a cascade on 3's decision when 4^g and 5's decisions agree with 3's decision and (iii.b) 6^g follows her signal if either (i') 4^g and 5's decisions are the same but disagree with 3's decision or (ii') 3 and 5's decisions are the same but disagree with 4^g's decision.

Analogue of Proposition 3. Using these lemmas we write the following analogue of Proposition 3.

Proposition OA2. Equilibrium Play with Multiple Common Players.
In equilibrium, if prior to the first common player:
POA2.1: Both groups are in an a (r) cascade, then the cascade continues, i.e., 3 and all subsequent players always choose a (r).
POA2.2: One group is in a cascade on a (r) and the other group is not in a cascade, then the cascade spills over, i.e., 3 and all subsequent players in both groups always choose a (r).
POA2.3: Neither group is in a cascade, then the common players follow their signals. If their decisions agree, then all players moving after 5 are in a cascade on her decision. If their decisions disagree, then both players moving after 5 follow their own signals. (Players 4^A and 4^B always cascade on 3's decision.)

POA2.4: Both groups are in opposing cascades, then 3 follows her signal and breaks one of the group's cascades, say B's, so 4^B follows her signal. If 3 and 4^B make the same decision, then 5 and all subsequent players are in a cascade on that decision. If 3 and 4^B make different decisions, then 5 follows her signal. When this happens, the cascade in A ends if 3 and 5 make different decisions and the cascade in A continues if 3 and 5 make the same decision; in either event, 6^B follows her signal. The analogous result obtains if 3 breaks A's cascade.

An Anonymous Common Player

In this subsection, we consider an example with an "anonymous" common player, i.e., one whose position is unknown before play begins, and we characterize equilibrium play.

Setup. There are two groups of six players that share a common player. Before play begins, nature randomly determines the common player's position. With equal probability, the common player is either in position three or position five. The common player knows her position, but the other players do not. The arrangement of players is as illustrated in Figure OA3. We assume $p = 2/3$. All other features of the environment are as in the main text.

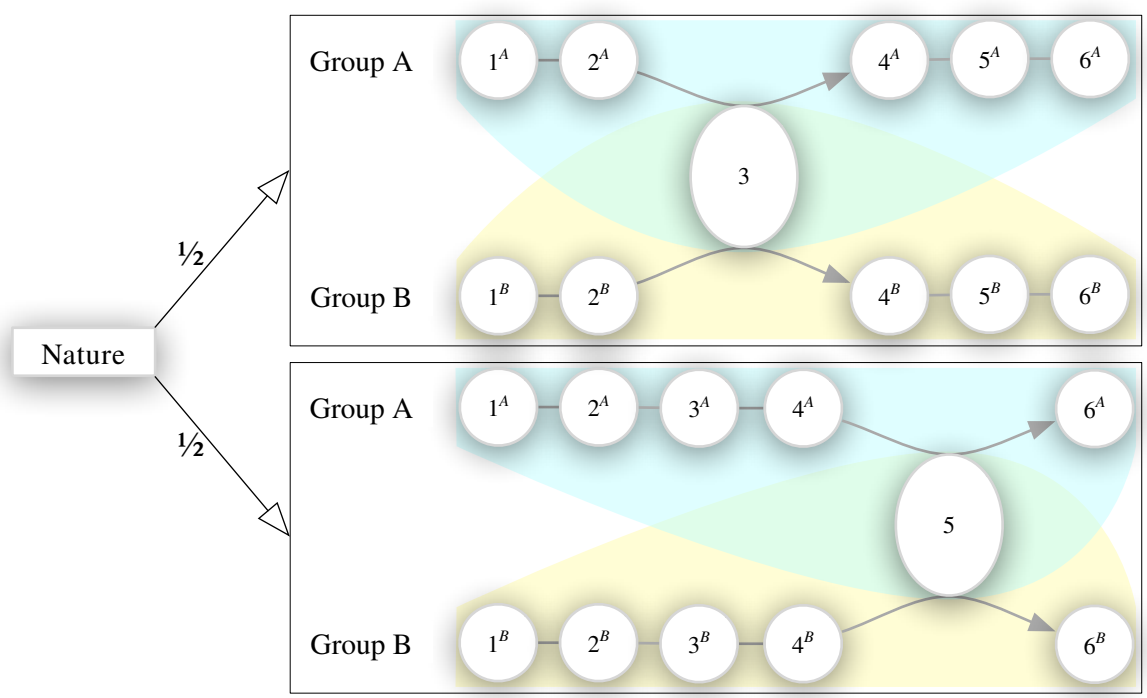

Figure OA3: Anonymous Common Player

Equilibrium. Since players 1^g and 2^g move before the common player, they behave exactly as in Proposition 1. If player 3^g is the common player, then her behavior is given by Proposition 2; otherwise, player 3^g behaves as in Proposition 1.

Ex ante, player 4^g believes there is a half chance that 3^g is the common player and a half chance that 3^g is a regular player. Ex post, however, the decisions of her predecessors may "reveal" whether 3^g is the common player – e.g., $(d_1^g, d_2^g, d_3^g) = (a, a, r)$ reveals that the common player is player 3^g, while $(d_1^g, d_2^g, d_3^g) = (a, a, a)$ does not. Taking account of this fact, we compute the equilibrium play of 4^g.

Lemma OA8. Equilibrium Play of Player 4^g.
If group g is not in a cascade (before player 3), then player 4^g makes the same decision as player 3^g. If group g is in a cascade and 3^g's decision agrees with the cascade, then 4^g makes the same decision as 3^g. If group g is in a cascade and 3^g's decision disagrees with the cascade, then 4^g follows her own signal.

In other words, 4^g behaves the same player $k+1^g$ in an interacting cascade. Intuitively, this occurs because 4^g assigns a high likelihood to 3^g being the common player.

Player 5^g knows whether she is the common player. If she is not, then she behaves as does player $k+2^g$ in the baseline model in light of Lemma OA8. If 5^g is the common player, then she either observes (a) the same cascade in both groups or (b) different cascades in both groups (since 4^g always imitates 3^g). If (a), then 5^g follows the cascade. If (b), then 5^g follows her signal. The next lemma summarizes.

Lemma OA9. Equilibrium Play of Player 5^g.
(i) If player 5^g is not the common player, then (i.a) 5^g makes the same decision as player 3 if her group is not in a cascade (before 3), (i.b) 5^g makes the same decision as player 3 if her group is in a cascade and 3's decision agrees with the cascade, (i.c) 5^g makes the same decision as player 3 if her group is in a cascade that disagrees with the decision of 3 and players 3 and 4^g make the same decision, (i.d) 5^g follows her signal if her group is in a cascade that disagrees with the decision of 3 and players 3 and 4^g make different decisions. (ii) If player 5^g is the common player, then (ii.a) she makes the same decision as her predecessors when both groups are in cascades on the same decision and (ii.b) she follows her signal when both groups are in cascades on different decisions.

The next lemma characterizes 6^g's behavior.

Lemma OA10. Equilibrium Play of Player 6^g.

(i) If there is a cascade in g (before to player 3^g) then, (i.a) 6^g follows the cascade if players 3^g and 5^g both follow the cascade, (i.b) 6^g follows her signal if 3^g follows the cascade and 5^g does not follow the cascade, (i.c) 6^g follows her signal if 3^g does not follow the cascade and 5^g makes a different decision than 3^g, and (i.d) 6^g makes the same decision as 3^g if 3^g does not follow the cascade and 5^g makes the same decision as 3^g.

(ii) If there is no cascade, then (ii.a) 6^g makes the decision as 3^g if players 3^g and 5^g make the same decision and (ii.b) 6^g follows her own signal otherwise.

Analogue of Proposition 3. We use these lemmas to write the following analogue of Proposition 3.

Proposition OA3. Equilibrium Play with an Anonymous Common Player.

If the common player is in the third position, then play is described by Proposition 3. If the common player is in the fifth position, she always observes a cascade in both groups. If these cascades are the same, then she follows them and the cascades continue. If these cascades differ, then she follows her own signal. The cascade that agrees with her decision continues and the cascade that disagrees breaks.